Vector Derivative Identities and Properties

$$f = f(x,y,z) = f(\mathbf{x}) \qquad \mathbf{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

$$\mathbf{F}(x,y,z) = P(x,y,z)\hat{\mathbf{i}} + Q(x,y,z)\hat{\mathbf{j}} + R(x,y,z)\hat{\mathbf{k}}$$

$$\nabla = \hat{\mathbf{i}}\partial_x + \hat{\mathbf{j}}\partial_y + \hat{\mathbf{k}}\partial_z$$

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \qquad \text{(vector)}$$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \qquad \text{(scalar)}$$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} \qquad \text{(vector)}$$

$\mathbf{F}, \mathbf{G}$: vector fields, f: scalar function, ψ: either

1. Orthogonality of the gradient to level curves or surfaces: $\nabla f(\mathbf{x}_0) \perp \{f(\mathbf{x}) = f_0\}$ at $\mathbf{x}_0$
2. Vector forms of the product rule:

$$\nabla \cdot (f\mathbf{F}) = \mathbf{F} \cdot \nabla f + f\, \nabla \cdot \mathbf{F} \tag{1a}$$

$$\nabla \times (f\mathbf{F}) = (\nabla f) \times \mathbf{F} + f\,(\nabla \times \mathbf{F}) \tag{1b}$$

$$\nabla(\mathbf{F}\cdot\mathbf{G}) = \begin{array}{c} (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{G}\times(\nabla\times\mathbf{F}) \\ +\,(\mathbf{F}\cdot\nabla)\mathbf{G} + \mathbf{F}\times(\nabla\times\mathbf{G}) \end{array} \tag{1c}$$

$$\nabla \cdot (\mathbf{F}\times\mathbf{G}) = (\nabla\times\mathbf{F})\cdot\mathbf{G} - \mathbf{F}\cdot(\nabla\times\mathbf{G}) \tag{1d}$$

$$\nabla \times (\mathbf{F}\times\mathbf{G}) = \begin{array}{c} (\mathbf{G}\cdot\nabla)\mathbf{F} - (\nabla\cdot\mathbf{F})\mathbf{G} \\ -\,(\mathbf{F}\cdot\nabla)\mathbf{G} + (\nabla\cdot\mathbf{G})\mathbf{F} \end{array} \tag{1e}$$

$$\nabla \cdot (\mathbf{F}\mathbf{G}) = (\nabla\cdot\mathbf{F})\mathbf{G} + (\mathbf{F}\cdot\nabla)\mathbf{G} \tag{1f}$$

3. The curl-free property of gradient fields ($\forall f$): $\boxed{\nabla \times \nabla f = \mathbf{0}}$
4. The divergence-free property of curl fields ($\forall \mathbf{F}$): $\boxed{\nabla \cdot (\nabla \times \mathbf{F}) = 0}$
5. div(grad)=Laplacian: $\boxed{\nabla \cdot \nabla \psi = \nabla^2 \psi} \equiv \Delta\psi$
6. curl-curl-grad(div)-Laplacian: $\boxed{\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}}$

Integral Theorems of Vector Calculus

Green's theorem in the xy plane

For region D inside closed-curve C with no singularities of P, Q inside C:

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

If singularities are present, then the values of the line and double integral are not promised to be equal.

2D Vector versions of Green's theorem

In the xy plane, with $\mathbf{F} = (P(x,y), Q(x,y), 0)$:

- **2D Stokes' theorem** (**Green's Work Theorem**)

$$\text{Work} = \boxed{\oint_C \mathbf{F}\cdot\mathbf{t}\,ds = \iint_D (\nabla\times\mathbf{F})\cdot\hat{\mathbf{k}}\,dA}$$

unit tangent vector, $\mathbf{t} = (x'(t), y'(t))/|\mathbf{x}'(t)|$

- **2D Divergence theorem** (**Green's Flux Theorem**)

$$\text{Flux} = \boxed{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \nabla\cdot\mathbf{F}\,dA}$$

unit outward normal vector, $\mathbf{n} = (y'(t), -x'(t))/|\mathbf{x}'(t)|$

$$\oint_C -Q\,dx + P\,dy = \iint_D (P_x + Q_y)\,dA$$

3D Stokes' theorem: Surface S with edge curve C

$$\text{Work} = \boxed{\oint_C \mathbf{F}\cdot\mathbf{t}\,ds = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS}$$

Surface integral: $\mathbf{n}$ is the "right-thumb" unit normal to S with edge curve C with fingers gripped in the direction of C

3D Divergence theorem: Volume D enclosed by closed surface S

$$\text{Flux} = \boxed{\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \nabla\cdot\mathbf{F}\,dV}$$

Line integrals on parametric curve $C = \{t : a \to b\}$

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

unit tangent $\mathbf{t} = \mathbf{x}'(t)/|\mathbf{x}'(t)|$, arclength $ds = |\mathbf{x}'(t)|\,dt$

$$\text{Work} = \int_C \mathbf{F}\cdot\mathbf{t}\,ds = \int_C \mathbf{F}\cdot d\mathbf{x}$$
$$= \int_a^b \mathbf{F}(\mathbf{x}(t))\cdot\mathbf{x}'(t)\,dt = \int_C P\,dx + Q\,dy + R\,dz$$

Surface integrals

$$\text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S \mathbf{F}\cdot d\mathbf{S}$$

Parametric curves

Position

$$\mathbf{x}(t) = (x(t), y(t), z(t))$$

Velocity

$$\mathbf{v}(t) = \frac{d\mathbf{x}}{dt} = (x'(t), y'(t), z'(t))$$

unit **Tangent** vector

$$\mathbf{t} = \hat{\mathbf{v}} = \frac{1}{|\mathbf{v}(t)|}\mathbf{v}(t)$$

In 2D: unit tangent and normal $\mathbf{t} \cdot \mathbf{n} = 0$

$$\mathbf{t} = \frac{(x'(t), y'(t))}{\sqrt{x'(t)^2 + y'(t)^2}} \qquad \mathbf{n} = \frac{(y'(t), -x'(t))}{\sqrt{x'(t)^2 + y'(t)^2}}$$

Product rules applied to parametric curves $\mathbf{x} = \mathbf{p}(t)$, $\mathbf{x} = \mathbf{q}(t)$ and scalar function $k(t)$

$$\frac{d}{dt}(k\mathbf{p}) = \frac{dk}{dt}\mathbf{p} + k\frac{d\mathbf{p}}{dt} \tag{2a}$$

$$\frac{d}{dt}(\mathbf{p} \cdot \mathbf{q}) = \frac{d\mathbf{p}}{dt} \cdot \mathbf{q} + \mathbf{p} \cdot \frac{d\mathbf{q}}{dt} \tag{2b}$$

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{q}) = \frac{d\mathbf{p}}{dt} \times \mathbf{q} + \mathbf{p} \times \frac{d\mathbf{q}}{dt} \tag{2c}$$

Surface integrals: list of unit normal vectors n

Parametric surface: $x = x(u, v), y = y(u, v), z = z(u, v)$

$$\mathbf{N} = \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \qquad \longrightarrow \qquad \mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|}$$

Graph of fcn: $z = g(x, y)$

$$\mathbf{n} = \frac{(-g_x, -g_y, 1)}{\sqrt{1 + g_x^2 + g_y^2}}$$

Plane: $ax + by + cz = d$

$$\mathbf{n} = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}}$$

Sphere: $\rho = a$

$$\mathbf{n} = \frac{1}{a}(x, y, z)$$

Circular cylinder: $r = a$

$$\mathbf{n} = \frac{1}{a}(x, y, 0)$$
