

Separation of Variables (V 2.0): Inhomogeneous PDE IBVP

- We need to determine the right choice of eigenfunction expansion for the problem:
 - Homogenize the PDE equation and the BC's.
 - Plug-in the trial solution $u_k(x, t) = a_k(t)\phi_k(x)$ and separate variables.
 - We will only consider PDE problems leading to Sturm-Liouville BVP problems, $L\phi_k = -\lambda_k\sigma\phi_k$. Identify the p, q, σ for the problem and make use of the general solution or known eigenfunctions.
 - Obtain the eigenfunctions of the x -BVPs: $\{\phi_k(x), \lambda_k\}$. **Stop.**¹
- Now we will use the $\phi_k(x)$'s to construct the solution in the form

$$u(x, t) = \sum_k b_k(t)\phi_k(x) \quad b_k(t) = \frac{\langle u, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle}$$

by determining equations for the $b_k(t)$ coefficient functions (t -ODEs for each k). For each k , work out the inner product equations:

$$\langle \text{PDE}, \phi_k \rangle \quad \text{and} \quad \langle \text{IC's}, \phi_k \rangle$$

ICs for u will produce ICs for b_k while BCs and inhomogeneous terms in the PDE will add inhomogeneous forcing to the the ODE for $b_k(t)$.

1 Useful results

1.1 Green's formula

If L is a Sturm-Liouville operator, $Lu \equiv \frac{d}{dx} \left(p(x) \frac{du}{dx} \right) + q(x)u$, with $L\phi = -\lambda\sigma\phi$ then

Green's formula gives the result for $\langle v, Lu \rangle = \langle u, Lv \rangle + (\text{Boundary terms})$:

$$\int_a^b vLu \, dx = p(x) \left(v \frac{du}{dx} - u \frac{dv}{dx} \right) \Big|_a^b + \int_a^b uLv \, dx$$

See Haberman (8.4.11), page 361. Using this formula saves you the work of doing the two levels of integration-by-parts usually involved in the adjoint calculation to get the boundary terms.²

1.2 Physical meanings of the boundary conditions

See Haberman's table 2.4.1 on page 162 for the physical interpretation of the different types of boundary conditions.

¹No need for $a_k(t)$ solutions...

²This formula is derived by just doing the IBP² with the general SL L operator.

1.3 $p = 1, q = 0, \sigma = 1$ eigensolutions

This is my version of Haberman's very useful table from p. 69.³

$$\text{Solutions of } \boxed{\phi_k''(x) = -\lambda_k \phi_k(x) \quad \text{on} \quad 0 \leq x \leq 1}$$

with various common boundary conditions:

Boundary Conditions	$\phi_k(0) = 0,$ $\phi_k(1) = 0,$ Dirichlet/Dirichlet	$\phi_k'(0) = 0,$ $\phi_k'(1) = 0,$ Neumann/Neumann	$\phi_k(0) = 0,$ $\phi_k'(1) = 0,$ Dirichlet/Neumann
Indices $k =$	$1, 2, 3, \dots$	$0, 1, 2, 3, \dots$	$0, 1, 2, 3, \dots$
Eigenvalues $\lambda_k =$	$k^2 \pi^2$	$k^2 \pi^2$	$\left(\frac{2k+1}{2}\right)^2 \pi^2$
Eigenfns $\phi_k(x) =$	$\sin(k\pi x)$	$\cos(k\pi x)$	$\sin\left(\frac{2k+1}{2}\pi x\right)$

- To swap $x = 0$ and $x = 1$ BC's: replace $x \rightarrow (1 - x)$ in $\phi_k(x)$.
- Haberman also gives the solutions for periodic BC's (the full Fourier Series).
- Robin boundary conditions for this equation yield problems where the λ_k 's must be calculated numerically...

1.4 Reminder: solving linear inhomogeneous ODE IVP

$$\frac{db}{dt} + r(t)b = I(t) \quad b(0) = b_0 \quad (1)$$

The integrating factor approach is equivalent to the variation-of-parameters method for first order inhomogeneous linear equations: first solve the homogeneous equation

$$\frac{db_h}{dt} + r(t)b_h = 0 \quad \rightarrow \quad b_h(t) = e^{-\int r(t) dt}$$

Then substitute $b(t) = c(t)b_h(t)$ into (1) to obtain an equation for dc/dt

$$b(t) = c(t)e^{-\int r(t) dt} \quad \rightarrow \quad \frac{dc}{dt} = I(t)e^{\int r(t) dt}$$

and integrate to obtain $c(t)$.

Then finally substitute back and impose the IC to obtain the solution $b(t)$.

Simplest case: if $r = \text{constant}$, then

$$b(t) = c(t)e^{-rt} \quad c(t) = \int_0^t I(t')e^{rt'} dt' + C \quad \rightarrow \quad b(t) = \left(\int_0^t I(t')e^{rt'} dt' + b_0 \right) e^{-rt}$$

³This is the first in a series of sets of solutions of different Sturm-Liouville problems for different (p, q, σ) subject to different boundary conditions. Its good to have these results handy rather than using time to re-derive them from scratch.