Green's functions for ODE BVP: supplementary notes

1. The ODE BVP – Linear algebra analogy

An $n \times n$ linear algebra problem and the formula for its solution:

$$\mathbf{L}\vec{\mathbf{u}} = \vec{\mathbf{f}} \qquad\qquad \vec{\mathbf{u}} = \mathbf{L}^{-1}\vec{\mathbf{f}} \tag{1}$$

An analogous ODE BVP and the Green's function integral representation of its solution:

$$Lu = f(x) \qquad u(x) = \int_a^b g(x,s)f(s)\,ds \tag{2}$$

Lets call the solution integral operator M,

$$Mv(x) = \int_a^b g(x,s)v(s)\,ds.$$

- The solution of the linear algebra is computed by multiplying the inverse matrix on the RHS vector, $\vec{\mathbf{u}} = \mathbf{L}^{-1}\vec{\mathbf{f}}$.
- The solution of the ODE is computed by computing the integral with the Green's function kernel on the RHS function, $u(x) = Mf$.
- These two things match if M is the inverse operator to L – i.e. M undoes whatever L does. Test this idea by applying M to both sides of $Lu = f$:

$$M(Lu = f) \quad \to \quad (ML)u = Mf \quad \to \quad 1 \cdot u = Mf \quad \to \quad u(x) = Mf$$

This is true!

- The Green's function integral is the inverse operator to the LHS linear differential operator L.

This is a generalization of the idea that the integral is the "anti-derivative" – the opposite operation to taking a derivative: $\int \frac{d}{dx}u(x)\,dx = u(x)$.

The Green's function integral is the inverse operation to the differential equation.

2. The physics of "Point sources"

Suppose that the RHS forcing function is just at one point $f(x) = f_0\delta(x - x_0)$ then the solution is

$$u(x) = \int_a^b g(x,s)f_0\delta(s - x_0)\,ds = \boxed{f_0 \cdot g(x,x_0) = u(x)}$$

Here f_0 is the forcing amplitude and $g(x,x_0)$ is the "point source solution" which describes how the influence from the forcing at point x_0 is felt by the solution at point x.

What is the linear algebra interpretation?
Call $\mathbf{M} = \mathbf{L}^{-1}$. Then the solution is $\vec{\mathbf{u}} = \mathbf{M}\vec{\mathbf{f}}$.
Suppose only the 7^{th} element of $\vec{\mathbf{f}}$ is nonzero, $\vec{\mathbf{f}}^T = (0,0,0,\cdots,0,f_7,0,\cdots,0,0,0)^T$. Then the matrix-vector product will zero-out everything except the 7^{th} column of $\mathbf{M}$:
$\vec{\mathbf{u}}^T = f_7\,([7^{th}\text{ column vector of }\mathbf{M}])^T$.
Call $\vec{\mathbf{w}}_k$ the k^{th} column vector of $\mathbf{M}$, then the general solution due to a $\vec{\mathbf{f}}$ with all entries is

$$\vec{\mathbf{u}} = \sum_{k=1}^{n} f_k\vec{\mathbf{w}}_k \tag{3}$$

The analogy is (3) is like (2) since: $\sum_k$ is like $\int ds$, k is like s, and $\vec{\mathbf{w}}_k$ is like $g(x,s)$.