Math 211, Fall 2009, Duke University Lecture 14

Math 211 Part II: Separation of Variables for solving PDEs

Example 1: IBVP for the heat equation for $u(x,t)$ on $0 \le x \le 1$ with $t \ge 0$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \underbrace{u(x=0)=0, \quad u(x=1)=0}_{BC's} \qquad \underbrace{u(t=0)=f(x)}_{IC} \quad (1)$$

The Separation of Variables Solution Process (v1.0) [for Homogeneous BC's]

Goal: To write soln of the full problem via linear superposition as

$$u(x,t) = \sum_k a_k(t)\phi_k(x)$$

where $\phi_k(x)$ are orthogonal[a] functions to be determined

- Divide and Conquer: Can construct the solution if we require that each product $u_k(x,t) = a_k(t)\phi_k(x)$ is separately a nontrivial solution of the homogenized[b] version of the problem

$$\frac{\partial u_k}{\partial t} = \frac{\partial^2 u_k}{\partial x^2} \qquad u_k(x=0)=0, \quad u_k(x=1)=0 \qquad \underbrace{u_k(t=0)=c_k\phi_k(x)}_{\Downarrow}$$

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[a] Either bi-orthogonal $\langle \psi_j, \phi_k \rangle$ or self-orthogonal $\langle \phi_j, \phi_k \rangle_\sigma$

[b] Drop all inhomogeneous terms from the PDE and the BC's, but IC's are treated differently

Why are IC's different?

- BVP (Boundary Val Probs) for linear homogeneous eqns with zero BC's can have nontrivial solutions as eigenfunctions $\phi_k(x)$ [oscillatory solutions]
- IVP (Initial Val Probs) for linear homogeneous eqns with all zero IC's produce only the trivial soln $a_k(t) \equiv 0$ [zero at $t = 0$ stays dead for $t > 0$] (No good)

- Substitute-in $u_k(x,t) = a_k(t)\phi_k(x)$ into all parts of the problem

$$\text{PDE}: \qquad \frac{da_k}{dt}\phi_k = a_k \frac{d^2\phi_k}{dx^2}$$

$$\text{BC}_1: \qquad a_k(t)\phi_k(0) = 0$$

$$\text{BC}_2: \qquad a_k(t)\phi_k(1) = 0$$

$$\text{IC}: \qquad a_k(0)\phi_k(x) = c_k\phi_k(x)$$

BC's must be true for all $t \geq 0$ $\Longrightarrow$ $\phi_k(0) = 0 \qquad \phi_k(1) = 0$

IC must be true for all $0 \leq x \leq 1$ $\Longrightarrow$ $a_k(0) = c_k$

PDE must be true for all x and t $\Longrightarrow$ $\boxed{\text{separate } x, t\text{-dependence:}}$

$$\underbrace{\frac{1}{a_k(t)}\frac{da_k}{dt}}_{t\text{'s only}} = \underbrace{\frac{1}{\phi_k(x)}\frac{d^2\phi_k}{dx^2}}_{x\text{'s only}}$$

"Separated form"

The only way $T_k(t) = X_k(x)$ for all (x, t) is if

$$T_k(t) = X_k(x) = S_k = \text{constant}$$

"S_k separation constant" for each k: unknown, to be determined

- Separated t-problem: ODE IVP

$$\frac{1}{a_k}\frac{da_k}{dt} = S_k \quad \rightarrow \quad \left\{ \begin{array}{l} \dfrac{da_k}{dt} = S_k a_k \\ a_k(0) = c_k \end{array} \right\} \quad \rightarrow \quad \boxed{a_k(t) = c_k e^{S_k t}}$$

IVP works with any S_k, values not determined yet.

- Separated x-problem: ODE BVP

$$\frac{1}{\phi_k}\frac{d^2\phi_k}{dx^2} = S_k \quad \rightarrow \quad \left\{ \begin{array}{l} \phi_k'' = S_k \phi_k \\ \phi_k(0) = 0 \\ \phi_k(1) = 0 \end{array} \right\}$$

To match eigenvalue BVP problems, relabel $S_k = -\lambda_k$:

$$\phi_k'' + \lambda_k \phi_k = 0 \qquad \phi_k(0) = \phi_k(1) = 0$$

$$\boxed{\phi_k(x) = \sin(k\pi x) \qquad S_k = -\lambda_k = -k^2\pi^2 \qquad k = 1, 2, 3\ldots}$$

Self-adjoint problem: completeness, orthogonality, eigenvalue results...

• Everything pinned down, except IC $f(x)$
Final solution via linear combination of u_k solns

$$u(x,t) = \sum_k u_k(x,t) = \sum_{k=1}^{\infty} c_k e^{-k^2\pi^2 t} \sin(k\pi x)$$

Need c_k's to match IC:

$$u(t=0) = \sum_{k=1}^{\infty} c_k \sin(k\pi x) = f(x) \qquad \boxed{c_k = \frac{\langle f, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle}}$$

Overall: PDE solution is an eigenfunction expansion ($\phi_k(x)$ from the x-BVP) with coefficients that depend on the IC and on the solutions of the t-IVP ■

Further notes:

- The successes of separation of variables:

  The 3 fundamental PDE problems

- Examples of problems where separation of variables does not work

Three fundamental classes of second order PDEs

1. **Parabolic**: "diffusive spreading"

   basic equation: the heat equation (and inhomogeneous versions)[a]

   $$\boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}} \qquad \longrightarrow \qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

   Separation of variables: (x-BVP)×(t-IVP)=2 BC's (1 on each end), 1 IC

2. **Hyperbolic**: "wave propagation"

   basic equation: the wave equation

   $$\boxed{\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}} \qquad \longrightarrow \qquad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

   Separation of variables: (x-BVP)×(t-IVP)=2 BC's (1 on each end), 2 IC's

3. **Elliptic**: "equilibrium states"

   basic equation: Laplace's equation → Poisson's equation

   $$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \qquad \longrightarrow \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

   Separation of variables: (x-BVP)×(y-BVP)=(2 BC's (1 on each end))^2

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[a] Inhomogeneous terms often called "source terms"

Well-posed problems: PDE and appropriate side conditions (IC's and BC's) to produce a unique solution

Three classes of side-conditions

1. **Dirichlet**: sets value of solutions on boundary

   $$u(x=0,t)=A$$

   More generally, value can be a function of the other variable,

   $$u(x=0,t)=A(t)$$

2. **Neumann**: sets value of derivative (∂ in direction thru the boundary) ("flux")

   $$\left.\frac{\partial u}{\partial x}\right|_{x=0}=B \qquad \text{or} \qquad \left.\frac{\partial u}{\partial y}\right|_{y=0}=C$$

   and more generally $u_x(x=0,t)=B(t)$ or $u_y(x,y=0)=C(x)$

3. **Robin**: sets a linear combination of Dirichlet and Neumann at boundary

   $$\text{at } x=0: \qquad \frac{\partial u}{\partial x}+Du=E$$

Examples of non-separable problems: where separation-of-variables is not possible

- (BC issue) A time-dependent Robin BC at $x = 0$

$$\frac{\partial u}{\partial x} + D(t)u = E(t)$$

homogenize

$$u_x + D(t)u = 0$$

substitute-in $u_k(x,t) = a_k(t)\phi_k(x)$

$$a_k(t)\left[\phi_k'(0) + D(t)\phi_k(0)\right] = 0$$

- (PDE issue) A time-dependent convection-diffusion equation

$$u_t + C(t)u_x = u_{xx} + Q(x,t)$$

homogenize

$$u_t + C(t)u_x = u_{xx}$$

substitute-in $u_k(x,t) = a_k(t)\phi_k(x)$ and try to separate variables

$$\frac{1}{a_k(t)}\frac{da_k}{dt} + \frac{C(t)}{\phi_k(x)}\frac{d\phi_k}{dx} = \frac{1}{\phi_k(x)}\frac{d^2\phi_k}{dx^2}$$