Green's functions for ODE BVP (Haberman, section 9.3)

- Bilinear form of $g(x, t)$ from eigenfunction expansion is an infinite series
- Piecewise-defined $g(x, t)$ from variation of parameters seems a bit tedious
- An alternative (faster & more clever) theory for building the piecewise $g(x, t)$:

Generalized functions and distribution theory

- Very un-usual "functions" - some bad properties (unbounded, not smooth, ...) but also have special "super-powers" that normal functions can't do
- Super-powers used in inner products $\langle f(x), g(x) \rangle$ and provide massive short-cuts to long calculations
- Two powerful distributions for math/physics/engineering:

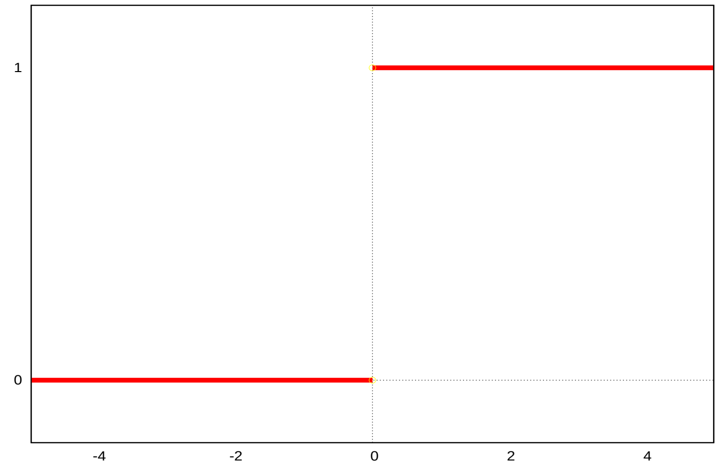
  The Heaviside Step fcn: $H(x)$ and **The Dirac Delta fcn**: $\delta(x)$

Oliver Heaviside (1850-1925) British electrical engineer (fiery temper)

Paul Dirac (1902-1984) British physicist (a god of quantum mechanics)

The Heaviside step function $H(x)$

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$



- Discontinuous jump at $x = 0$
- $H(0)$? Arguments about $= 0$ or $= 1$ or $= \frac{1}{2}$ or ... or "not important"...
- Used as a "switch" to turn other functions on/off:

$$f(x)H(x - x_*) = \begin{cases} 0 & x < x_* \\ f(x) & x > x_* \end{cases}$$

- Chops-off range of integration:

$$\langle f(x), H(x - x_*)\rangle = \int_{-\infty}^{\infty} f(x)H(x - x_*)\, dx = \int_{x_*}^{\infty} f(x)\, dx$$

The Heaviside step function $H(x)$ (cont)

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

- Derivative (for integration by parts...)

$$\frac{dH}{dx} = \begin{cases} 0 & x < 0 \\ 0 & x > 0 \\ ? & x = 0 \end{cases}$$

Try using the limit definition of derivative

$$H'(0) = \lim_{\epsilon \longrightarrow 0} \frac{H(\epsilon) - H(0)}{\epsilon} \;? \qquad H'(0) = \lim_{\epsilon \longrightarrow 0} \frac{H(0) - H(-\epsilon)}{-\epsilon} \;?$$

Average them :)

$$H'(0) = \lim_{\epsilon \longrightarrow 0} \frac{H(\epsilon) - H(-\epsilon)}{2\epsilon} = \frac{1-0}{2\epsilon} = \infty!?!$$

The Dirac Delta function $\delta(x)$

- Definition by relation to $H(x)$: $\boxed{\delta(x) = H'(x)}$
- "Definition" via values: $\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$
- Actual "operational meaning" (Super-power, i.e. what does it do?):

$$\boxed{\int_{-\infty}^{\infty} f(x)\delta(x - x_*)\, dx = f(x_*)} \qquad \forall \text{ nice } f(x)$$

called the *"sifting property"*. Proof via integration by parts:

$$\begin{aligned}
\int_{-\infty}^{\infty} \underbrace{f(x)}_{u}\, \underbrace{H'(x - x_*)\, dx}_{dv} &= f(x)H(x - x_*)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)H(x - x_*)\, dx \\
&= \Big(f(\infty) - 0\Big) - \int_{x_*}^{\infty} f'(x)\, dx \\
&= f(\infty) - \Big(f(x)\big|_{x_*}^{\infty}\Big) \\
&= f(x_*)
\end{aligned}$$

$\int_{-\infty}^{\infty}$ can be any $\int_a^b$ with $a < x_* < b$