Variation of Parameters for Inhomogeneous BVP

Second order problems: general inhomogeneous equation in standard form[1]

$$\frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)u = f(x) \qquad a \le x \le b \tag{1a}$$

with one typical set of homogeneous BCs[2] (all others sets of homogeneous BCs work similarly)

$$u'(a) = 0 \qquad u(b) = 0 \tag{1b}$$

Solution process steps:

1. Homogenize the equation and solve:

$$u'' + Pu' + Qu = 0 \qquad \to \qquad \boxed{\text{two linearly independent solns } u = \{w_a(x), w_b(x)\}} \tag{2}$$

2. Use linear combinations of $w_a(x), w_b(x)$ to $\boxed{\text{make two solutions } u_1(x), u_2(x)}$ that satisfy the BC's: $\boxed{\text{One BC satisfied by each}}$: Pick A_1, A_2 to have $u_1(x)$ satisfy one BC:

$$u_1(x) = A_1 w_a(x) + A_2 w_b(x) \qquad \to \qquad u_1'(a) = A_1 w_a'(a) + A_2 w_b'(a) = 0 \tag{3a}$$

Pick B_1, B_2 to have $u_2(x)$ satisfy the other BC:

$$u_2(x) = B_1 w_a(x) + B_2 w_b(x) \qquad \to \qquad u_2(b) = B_1 w_a(b) + B_2 w_b(b) = 0 \tag{3b}$$

3. **The Variation of Parameters (VoP) form**

$$\boxed{u(x) = c_1(x)u_1(x) + c_2(x)u_2(x)} \tag{4}$$

If c_1, c_2 were constants, this would be $u_h = c_1u_1(x) + c_2u_2(x)$, the general solution of the homogeneous problem. But, via VoP, we can get the solution of the inhomogeneous problem by $\boxed{\text{finding the fcns } c_1(x), c_2(x).}$
Warning: You need $u_1(x), u_2(x)$. Using $w_a(x), w_b(x)$ will not work!
We will use formula (4) to solve

$$u'' + Pu' + Qu = f(x) \qquad u'(a) = 0 \qquad u(b) = 0, \tag{5}$$

by using the facts about u_1, u_2:

$$\{k = 1, 2\ :\quad u_k'' + Pu_k' + Qu_k = 0\ \} \qquad u_1'(a) = 0 \qquad u_2(b) = 0. \tag{6}$$

Equation (4) is always the starting point, then there are several steps:

(a) We need to evaluate derivatives of $u(x)$, so we use the product rule:

$$u'(x) = c_1(x)u_1'(x) + c_2(x)u_2'(x) + \left[c_1'(x)u_1(x) + c_2'(x)u_2(x)\right] \qquad \boxed{\text{Set } c_1'(x)u_1(x) + c_2'(x)u_2(x) = 0.} \tag{7}$$

We have the freedom to make this assumption, and it effectively yields the same derivative for u' as if c_1, c_2 were constants, $u' = c_1u_1' + c_2u_2'$. Then

$$u''(x) = (c_1(x)u_1'(x) + c_2(x)u_2'(x))' = c_1u_1'' + c_2u_2'' + c_1'u_1' + c_2'u_2' \tag{8}$$

[1] meaning that the highest derivative term has coefficient one
[2] Solutions for problems with inhomogeneous BCs will be added-in later

(b) Substitute into the equation

$$\begin{aligned} u'' + Pu' + Qu &= f(x) && \text{(9a)} \\ (c_1 u_1'' + c_2 u_2'' + c_1' u_1' + c_2' u_2') + P(c_1 u_1' + c_2 u_2') + Q(c_1 u_1 + c_2 u_2) &= && \text{(9b)} \\ c_1 \underbrace{(u_1'' + Pu_1' + Qu_1)}_{=0} + c_2 \underbrace{(u_2'' + Pu_2' + Qu_2)}_{=0} + c_1' u_1' + c_2' u_2' &= && \text{(9c)} \\ \boxed{c_1' u_1' + c_2' u_2'} &= \boxed{f(x)} && \text{(9d)} \end{aligned}$$

(c) In summary, we get a system of linear equations for the derivatives[3] of the c's

$$\begin{array}{l} u_1 c_1' + u_2 c_2' = 0 \\ u_1' c_1' + u_2' c_2' = f(x) \end{array} \quad \rightarrow \quad \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{10}$$

The determinant of the matrix of the u_k solutions and their derivatives is called the <u>**Wronskian**</u>

$$W(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} \tag{11}$$

The solution of this matrix system can be written out using <u>**Cramer's rule**</u>[4] in terms of determinants:

$$c_1' = \frac{\begin{vmatrix} 0 & u_2 \\ f & u_2' \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}} = -\frac{u_2 f}{W} \qquad c_2' = \frac{\begin{vmatrix} u_1 & 0 \\ u_1' & f \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}} = \frac{u_1 f}{W} \tag{12}$$

So we have differential equations for $c_1(x), c_2(x)$

$$\frac{dc_1}{dx} = -\frac{u_2(x) f(x)}{W(x)} \qquad \frac{dc_2}{dx} = \frac{u_1(x) f(x)}{W(x)} \tag{13}$$

(d) Use the homogeneous BCs on the u_k's to get ICs for the c_k's:

$$u'(a) = c_1(a) \underbrace{u_1'(a)}_{=0} + c_2(a) u_2'(a) = 0 \quad \rightarrow \quad c_2(a) = 0 \tag{14a}$$

$$u(b) = c_1(b) u_1(b) + c_2 \underbrace{u_2(b)}_{=0} = 0 \quad \rightarrow \quad c_1(b) = 0 \tag{14b}$$

Initial value problems and their solutions:

$$\frac{dc_1}{dx} = -\frac{u_2(x) f(x)}{W(x)} \qquad c_1(b) = 0 \quad \rightarrow \quad \boxed{c_1(x) = -\int_b^x \frac{u_2(t) f(t)}{W(t)}\, dt} \tag{15a}$$

$$\frac{dc_2}{dx} = \frac{u_1(x) f(x)}{W(x)} \qquad c_2(a) = 0 \quad \rightarrow \quad \boxed{c_2(x) = \int_a^x \frac{u_1(t) f(t)}{W(t)}\, dt} \tag{15b}$$

then the final solution is

$$\boxed{u(x) = c_1(x) u_1(x) + c_2(x) u_2(x).} \tag{16}$$

[3] 1st eqn from u' assumption, 2nd from substitution – only terms with the derivatives of the c_k's will appear.

[4] See basic math sheet