

**Fredholm Integral Equations:** supplementary notes

1. Eigenproblems,  $L\phi = \lambda\phi$ , for integral equations with separable kernels:

$$Lu \equiv \gamma u(x) + \int_a^b k(x,t)u(t) dt \quad k(x,t) = \sum_{j=1}^n \alpha_j(x)\beta_j(t) \tag{1}$$

For Fredholm integral equations of first kind (FIE<sub>1</sub>):

$$\int_a^b k(x,t)\phi(t) dt = \lambda\phi(x) \tag{2}$$

For  $\lambda \neq 0$ , use the trial solution for the eigenfunction

$$\phi(x) = \sum_{j=1}^n c_j \alpha_j(x), \tag{3}$$

to substitute into  $L\phi = \lambda\phi$  and then solve the resulting  $n \times n$  matrix eigenvalue problem to determine  $\lambda, \{c_j\}$ . These are the eigen-modes of finite multiplicity.

**Example** with  $n = 2$  and  $\alpha_1(x) = x, \alpha_2(x) = x^2$

$$\int_0^1 (4xt - 5x^2t^2)\phi(t) dt = \lambda\phi(x) \quad \begin{cases} \lambda_1 = \frac{1}{2} & \phi_1(x) = -6x + 5x^2 \\ \lambda_2 = -\frac{1}{6} & \phi_2(x) = 2x - 3x^2 \end{cases} \tag{4}$$

For FIE<sub>1</sub> there is a zero eigenvalue, where the eigenfunction is anything that satisfies the  $n$  orthogonality equations coming from  $L\phi_0 \equiv 0$ :

$$\int_a^b k(x,t)\phi_0(t) dt \equiv 0 \quad \rightarrow \quad \{ \langle \beta_j(t), \phi_0(t) \rangle = 0 \text{ for all } j = 1, 2, \dots, n \}. \tag{5}$$

There is an infinite set of such solutions  $\phi_0 = \phi_{\infty,m}(t)$  for  $m = 1, 2, 3, \dots, \infty$ .

This corresponds to  $\lambda = 0$  being an eigenvalue of infinite multiplicity  $\boxed{\lambda^\infty = 0}$ .

**Example** (continued) -  $\beta_1(t) = 4t, \beta_2(t) = -5t^2$ . The  $\phi_\infty(t)$  functions must satisfy:

$$\int_0^1 t\phi_\infty(t) dt = 0 \quad \text{and} \quad \int_0^1 t^2\phi_\infty(t) dt = 0. \tag{6}$$

Lots of  $\phi_\infty(t)$  can be worked out by plugging-in polynomials with two free coefficients, to be determined by these two equations, (for example  $\phi_{\infty,1}(t) = 1 + c_1t + c_2t^2$ ),

$$\phi_{\infty,1}(t) = 1 - 4t + \frac{10}{3}t^2, \quad \phi_{\infty,2}(t) = \frac{2}{5}t - \frac{4}{5}t^2 + t^3, \quad \dots \tag{7}$$

These  $\phi_\infty$  will be linearly independent functions, but they will not automatically be orthogonal. They can be made into an orthogonal set by using the Gram-Schmidt orthogonalization process.<sup>1</sup>

<sup>1</sup>Look up from Linear Algebra, or use this reminder: To construct an orthogonal set of vectors  $\{\vec{v}_k\}$  from an ordered set of linearly independent vectors  $\{\vec{u}_k\}$ , subtract-off from  $\vec{u}_k$  all of the projections of  $\vec{u}_k$  onto the previously generated  $\vec{v}_j$  ( $j = 0, \dots, k - 1$ ) vectors:

$$\begin{aligned} \vec{v}_0 &= \vec{u}_0 \\ \vec{v}_1 &= \vec{u}_1 - \frac{\langle \vec{u}_1, \vec{v}_0 \rangle}{\langle \vec{v}_0, \vec{v}_0 \rangle} \vec{v}_0 \\ \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_0 \rangle}{\langle \vec{v}_0, \vec{v}_0 \rangle} \vec{v}_0 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 \\ \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_0 \rangle}{\langle \vec{v}_0, \vec{v}_0 \rangle} \vec{v}_0 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2 \end{aligned}$$

and so on. Observe that  $\langle \vec{v}_k, \vec{v}_j \rangle = 0$  for any  $k \neq j$ .

## 2. Summary of solving $\boxed{Lu = f(x)}$ for **FIE<sub>1</sub>** and **FIE<sub>2</sub>**

Write the assumed solution as an eigenfunction expansion (with un-determined coefficients):

$$u(x) = \left( \sum_{j=1}^n c_j \phi_j(x) \right) + \left( \sum_{m=1}^{\infty} c_{\infty,m} \phi_{\infty,m}(x) \right) \quad (8)$$

The first sum is from the finite-multiplicity eigen-modes.

The second sum is from the eigenmodes for  $\lambda^\infty$ .

Plugging this  $u(x)$  into the LHS of  $Lu = f$  yields

$$Lu = \left( \sum_{j=1}^n \lambda_j c_j \phi_j(x) \right) + \lambda^\infty \left( \sum_{m=1}^{\infty} c_{\infty,m} \phi_{\infty,m}(x) \right). \quad (9)$$

- For **FIE<sub>1</sub>**, we have  $\lambda^\infty = 0$ , so the second sum drops out, and the first sum must match the RHS function:

$$\sum_{j=1}^n \lambda_j c_j \phi_j(x) = f(x) \quad (10)$$

Since the  $\phi_j$  are combinations of  $\alpha_j$ 's, the expansion for  $u(x)$  and this equation can be re-written as

$$\sum_{j=1}^n (\text{mix of } d\text{'s}) \alpha_j(x) = f(x) \quad \text{by plugging-in} \quad \boxed{u(x) = \sum_{j=1}^n d_j \alpha_j(x)} \quad (11)$$

The  $d_j$  coefficients can be determined by matching LHS vs. RHS one  $\alpha_j(x)$  at a time.

This can be a good shortcut since it eliminates the need to determine the  $\phi_j$  eigenfunctions.

- If  $f(x)$  matches a sum of the  $\alpha_j(x)$ 's, then a solution exists, and by FAT case  $A_2$  there are infinitely many solutions:

$$u(x) = \sum_{j=1}^n d_j \alpha_j(x) + \sum_{m=1}^{\infty} q_m \phi_{\infty,m}(x) \quad \text{for any choices of } q_m\text{'s} \quad (12)$$

- If  $f(x)$  does not match any possible sum of  $\alpha_j(x)$ 's, then no solution is possible by FAT case  $A_1$ .

This direct-substitution-and-matching approach is much easier than trying to calculate the solvability equation because there will be an infinite-multiplicity of adjoint eigenfunctions  $\psi_\infty(x)$  for  $\lambda^\infty = 0$ , and therefore an infinite number of solvability equations.

- For **FIE<sub>2</sub>**, if there is no zero eigenvalue, by FAT case (B), there will exist a unique solution for any possible  $f(x)$ . For this case it can be shown that the infinite-multiplicity sum will reproduce a multiple of the RHS function  $f(x)$ , so  $Lu = f(x)$  can be solved similarly by plugging-in a trial solution of the form

$$\boxed{u(x) = p f(x) + \sum_{j=1}^n d_j \alpha_j(x)} \quad (13)$$

and matching LHS vs. RHS to determine the constant  $p$  and the  $d_j$ 's.