# The Fredholm Alternative Theorem (FAT)

**Either**

**(A)** The homogeneous adjoint problem

$$L^*\psi_0 = 0 \qquad B_a^*\psi_0 = 0 \qquad B_b^*\psi_0 = 0 \tag{1}$$

has a **nontrivial** solution.

**Xor**[1]

**(B)** The inhomogeneous problem

$$Lu = f(x) \qquad B_a u = c \qquad B_b u = d \tag{2}$$

has a **unique** solution $u(x)$ for any choices of $f(x), c, d$.

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(a) The homogeneous adjoint problem $L^*\psi = 0$ corresponds to the adjoint eigenvalue problem $L^*\psi = -\lambda\psi$ for a zero eigenvalue $\boxed{\lambda_0 = 0.}$

(b) The alternative, case (B), automatically guarantees existence and uniqueness of the solution of (2) if (A) is not true.

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The Fredholm alternative is usually used in practice as a quick way to determine what to expect from solving problem (2) before working through the whole expansion.

It is used in this way:

- If $L$ (and hence $L^*$) has no zero eigenvalue, then the eigenfunction expansion process will work and will produce a unique solution with no difficulties – **Case (B)**, via

$$B_k - c_k\lambda_k\langle\phi_k, \psi_k\rangle = \langle f, \psi_k\rangle \quad \xrightarrow{\forall k} \quad \{c_k\} \quad \rightarrow \quad u(x) = \sum_k c_k\phi_k(x) \tag{3}$$

- If **Case (A)** holds, then there are two sub-cases for what happens to the solution of (2) based on the $\lambda_0 = 0$ equation: $B_0 - 0 = \langle f, \psi_0\rangle$. This equation is called the **solvability condition.**

($\mathbf{A_1}$) **No solution of (2) exists** if the solvability equation leads to a contradiction.

($\mathbf{A_2}$) **The solution of (2) is non-unique** if the solvability equation is consistent. Then solutions exist, but since the value of the $c_0$ coefficient does not get pinned down, any value may be selected for this coefficient. In contrast, the values for the $c_k$ for $k = 1, 2, 3, \cdots$ are uniquely determined, as in (3). So:

$$\text{Solutions of (2) are} \qquad u(x) = c_0\phi_0(x) + \sum_{k=1}^{\infty} c_k\phi_k(x) \qquad \text{with any value for } c_0. \tag{4}$$

- To determine between cases (**A**) vs. (**B**) check for a zero eigenvalue: $\boxed{\lambda = 0?}$

- To determine between cases ($\mathbf{A_1}$) vs. ($\mathbf{A_2}$), you must find the $\lambda_0 = 0$ adjoint eigenfunction $\boxed{\psi_0(x)}$: both the forcing $\langle f, \psi_0\rangle$ and the boundary terms $B_0$ involve $\psi_0$.

The Fredholm alternative is universal – it works the same way for matrices, ODE boundary value problems (Sturm-Liouville or General), integral equations, ..., everything*

[1]XOR: eXclusive OR - one case or the other exclusively, never both together, never neither.