Math 211, Fall 2009, Duke University — Lecture 7

Sturm-Liouville Theory

- The new main equation on a finite interval $a \le x \le b$

$$\underbrace{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi}_{L\phi} = -\lambda\sigma(x)\phi \qquad \to \qquad L\phi = -\lambda\sigma\phi \tag{1a}$$

  Assume $p(x), q(x), \sigma(x)$ are given real functions, and $\sigma(x) > 0$.

- The boundary conditions: general 'un-mixed' BC's,

$$A_1\phi(a) + A_2\phi'(a) = 0, \qquad B_1\phi(b) + B_2\phi'(b) = 0, \tag{1b}$$

  with $A_1^2 + A_2^2 > 0$ and $B_1^2 + B_2^2 > 0$ (i.e. not everything zeroed out).
  Un-mixed (also called 'separated') BC's means that the conditions at $x = a$ and $x = b$ do not include any terms from the other boundary: covers 1st, 2nd, 3rd kind BC's.

The Main Results

0. Reading: Haberman, section 5.3 (Sturm-Liouville Eigenvalue Problems) and 5.5 (Self-Adjoint Operators).

1. Any second-order eigenvalue problem

$$A(x)\frac{d^2\phi}{dx^2} + B(x)\frac{d\phi}{dx} + C(x)\phi = -\lambda\phi$$

   can be transformed into SL form (1a). BCs (1b) remain unchanged.
   Do-it-yourself proof by finding how $p(x), q(x), \sigma(x)$ relate to $A(x), B(x), C(x)$ – see Homework # 3, question 1.

2. $L^* = L$: the operator is formally self-adjoint in the usual $\langle f, g\rangle_{old} = \int_a^b fg\,dx$ inner product.

3. Equation (1a) with BCs (1b) is a self-adjoint problem, so

   (a) The eigenvalues are real.
   Proof by subtracting: $\langle \phi_k, L\overline{\phi_k}\rangle_{old} - \langle L^*\phi_k, \overline{\phi_k}\rangle_{old} = (\lambda_k - \overline{\lambda_k})||\phi_k||^2_{new} = 0$

   (b) Eigenfunctions for different eigenvalues are orthogonal in the σ-weighted inner product

$$\langle \phi_k, \phi_j\rangle_{new} = \int_a^b \phi_k(x)\phi_j(x)\sigma(x)\,dx = 0 \qquad j \neq k. \tag{2}$$

   Proof by subtracting: $\langle \phi_k, L\phi_j\rangle_{old} - \langle L^*\phi_k, \phi_j\rangle_{old} = (\lambda_k - \lambda_j)\langle \phi_k, \phi_j\rangle_{new} = 0$

4. **Regular SL problems**: with the additional assumption that

$$p(x) > 0 \qquad \text{on } a \le x \le b \tag{3}$$

   the problem gains the following properties (results of the Sturm-Liouville theorems):

   (a) There is a smallest eigenvalue, λ_1 (a finite number).

(b) The eigenvalues are discrete, distinct, and ordered real numbers:

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots \tag{4}$$

(i.e. there are no double, triple, etc. roots).

(c) The sequence of eigenvalues has $\lambda_k \to \infty$ as $k \to \infty$.

(d) There is a unique eigenfunction $\phi_k(x)$ for each λ_k.

(e) The eigenfunction $\phi_k(x)$ has $(k-1)$ zeros on $a < x < b$ (The oscillation theorem).[1]

(f) The eigenfunctions form a complete set for the expansion of any nice function

$$w(x) = \sum_{k=1}^{\infty} c_k \phi_k(x). \tag{5}$$

To take advantage of the orthogonality of the ϕ_k with respect to σ, we take inner products of both sides of the equation with respect to ϕ_j in $\langle , \rangle_{new}$:

$$c_j = \frac{\langle w, \phi_j \rangle_{new}}{\langle \phi_j, \phi_j \rangle_{new}} = \frac{1}{||\phi_j||^2_{new}} \int_a^b w(x)\phi_j(x)\sigma(x)\,dx \tag{6}$$

(g) <u>**But**</u> solving an inhomogeneous problem

$$Lu = f(x) \tag{7}$$

as an eigenfunction expansion

$$u(x) = \sum_{k=1}^{\infty} c_k \phi_k(x) \qquad c_k = \frac{\langle u, \phi_k \rangle_{new}}{\langle \phi_k, \phi_k \rangle_{new}} \tag{8}$$

<u>**works a little differently!**</u>
We still start by taking inner products of both sides of the equation with respect to ϕ_j, but now since the LHS will already introduce σ from $L\phi_j = -\lambda_j \sigma \phi_j$,
so <u>**we use the old inner product**</u> to get

$$\begin{aligned}
\langle Lu, \phi_j \rangle_{old} &= \langle f, \phi_j \rangle_{old} \\
B_j + \langle u, L^*\phi_j \rangle_{old} &= \\
B_j - \lambda_j \langle u, \sigma\phi_j \rangle_{old} &= \\
B_j - \lambda_j \langle u, \phi_j \rangle_{new} &= \\
B_j - \lambda_j c_j \langle \phi_j, \phi_j \rangle_{new} &= \langle f, \phi_j \rangle_{old} \\
\boxed{B_j - \lambda_j c_j \int_a^b \phi_j^2(x)\sigma(x)\,dx} &= \boxed{\int_a^b f(x)\phi_j(x)\,dx} \\
&= \int_a^b \left[\frac{f(x)}{\sigma(x)}\right] \phi_j(x)\sigma(x)\,dx \\
&= \langle [f/\sigma], \phi_j \rangle_{new}
\end{aligned}$$

So we see that we need to be careful with old vs. new inner products when we have a weight function: the adjoint operator is defined in terms of the old inner product, orthogonality is defined in the new inner product.

5. and ...

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[1] For proofs see <u>*Ordinary Differential Equations*</u> by Birkhoff and Rota, or other textbooks.