Eigenfunction expansions for Inhomogeneous ODE BVP

1 The General Inhomogeneous Second-Order Problem

$$\text{Inhomogeneous linear equation :} \qquad Lu(x) = f(x) \qquad a \le x \le b \tag{1a}$$

$$\text{Inhomogeneous boundary conditions :} \qquad \begin{array}{c} BC_1(a) = c \\ BC_2(b) = d \end{array} \tag{1b}$$

The Solution Process

1. "Homogenize" the problem: set all RHS's= 0

$$Lu = 0 \qquad BC_1(a) = 0 \qquad BC_2(b) = 0. \tag{2}$$

Then write the eigen-problem for this complete linear operator:

$$L\phi = -\lambda\phi \qquad BC_1(a) = 0 \qquad BC_2(b) = 0. \tag{3}$$

(a) Solve the eigenvalue problem for $\{\lambda_k, \phi_k(x)\}$.

(b) Determine the complete adjoint operator (assume real-inner-product)

$$L^*\psi = -\lambda\psi \qquad BC_1^*(a) = 0 \qquad BC_2^*(b) = 0. \tag{4}$$

(c) Determine the adjoint eigenfunctions $\{\psi_k(x)\}$.

2. Return to full problem and find the solution as the eigen-expansion:

$$u(x) = \sum_{k=1}^{\infty} c_k \phi_k(x) \qquad c_k = \frac{\langle \psi_k, u\rangle}{\langle \psi_k, \phi_k\rangle} \tag{5}$$

(a) Start by taking the inner product of both sides of ODE with the adjoint eigenfunctions

$$\langle \psi_k, Lu\rangle = \langle \psi_k, f\rangle \tag{6}$$

(b) Use Integration By Parts (IBP) on LHS to get to the adjoint. The IBP's will create Boundary Terms that will depend on ψ_k and the BC values c, d

$$B_k + \langle L^*\psi_k, u\rangle = \langle \psi_k, f\rangle \tag{7}$$

(c) Use the adjoint eigenvalue equation $L^*\psi_k = -\lambda_k\psi_k$

$$B_k - \lambda_k\langle \psi_k, u\rangle = \langle \psi_k, f\rangle \tag{8}$$

then use the expansion coefficient equation $\langle \psi_k, u\rangle = c_k\langle \psi_k, \phi_k\rangle$

$$B_k - \lambda_k c_k\langle \psi_k, \phi_k\rangle = \langle \psi_k, f\rangle \tag{9}$$

Solve for $\boxed{c_k}$ and then re-assemble to write $u(x)$ expansion as the final solution:

$$u(x) = \sum_{k=1}^{\infty} c_k \phi_k(x). \tag{10}$$

2 An example in detail

Solve for $u(x)$ on the interval $0 \le x \le 1$,

$$\frac{d^2u}{dx^2} = 9e^{4x}, \qquad u(0) = -5, \qquad u(1) = -7. \tag{11}$$

1. Homogenize the problem,

$$\frac{d^2u}{dx^2} = 0, \qquad u(0) = 0, \qquad u(1) = 0. \tag{12}$$

   (a) Find the solutions of the eigenvalue problem with the same L and same-kind boundary conditions,

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \qquad \phi(0) = 0, \qquad \phi(1) = 0. \tag{13a}$$

   This yields

$$\phi_k(x) = \sin(k\pi x), \qquad \lambda_k = k^2\pi^2, \qquad k = 1, 2, 3, \cdots \tag{13b}$$

   (b) Use integration by parts to determine the adjoint eigenvalue problem,

$$\frac{d^2\psi}{dx^2} = -\lambda\psi, \qquad \psi(0) = 0, \qquad \psi(1) = 0. \tag{14a}$$

   (c) The adjoint eigenfunctions are

$$\psi_k(x) = \sin(k\pi x), \qquad k = 1, 2, 3, \cdots \tag{14b}$$

2. Returning to the full problem, we seek an expansion for the solution in the form

$$u(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x), \qquad c_k = \frac{\langle u, \psi_k\rangle}{\langle \phi_k, \psi_k\rangle} = 2\int_0^1 u(x)\sin(k\pi x)\,dx \tag{15}$$

   (a) Take the inner product of the equation $u'' = f(x)$ with each adjoint eigenfunction, $k = 1, 2, \cdots$

$$\int_0^1 \underbrace{u''}\ \underbrace{\sin(k\pi x)\,dx} = \int_0^1 f(x)\sin(k\pi x)\,dx \tag{16}$$

   (b) Use IBP twice on LHS

$$(\psi_k u' - \psi_k' u)\Big|_0^1 + \int_0^1 \psi_k'' u\,dx = \int_0^1 f(x)\sin(k\pi x)\,dx \tag{17}$$

   (c) Use adjoint eigenvalue equation $\psi_k'' = -\lambda_k\psi_k$

$$(\psi_k u' - \psi_k' u)\Big|_0^1 - \lambda_k\int_0^1 \psi_k u\,dx = \int_0^1 f(x)\sin(k\pi x)\,dx \tag{18}$$

$$B_k - \lambda_k c_k\langle\psi_k, \phi_k\rangle = \langle\psi_k, f\rangle \tag{19}$$

   where B_k is the set of terms due to the boundary conditions. Solving (19) for c_k, we get

$$c_k = \frac{\langle\psi_k, f\rangle - B_k}{-\lambda_k\langle\psi_k, \phi_k\rangle}\ . \tag{20}$$

We are ready to evaluate everything, we observe a few key properties for the boundary terms,

$$\psi_k'(x) = k\pi\cos(k\pi x), \qquad \psi_k(0) = \psi_k(1) = 0 \qquad \psi_k'(0) = k\pi \qquad \psi_k'(1) = (-1)^k k\pi$$

and use the given information that $u(0) = -5$ and $u(1) = -7$.
Substituting-in for ϕ, ψ, λ and f and the BCs into (19),

$$-\psi_k'(1)u(1) + \psi_k'(0)u(0) - k^2\pi^2 c_k \int_0^1 \sin^2(k\pi x)\,dx = \int_0^1 9e^{4x}\sin(k\pi x)\,dx \tag{21a}$$

$$7(-1)^k k\pi - 5k\pi - \tfrac{1}{2}k^2\pi^2 c_k = -\frac{9k\pi(-1+(-1)^k e^4)}{16+k^2\pi^2} \tag{21b}$$

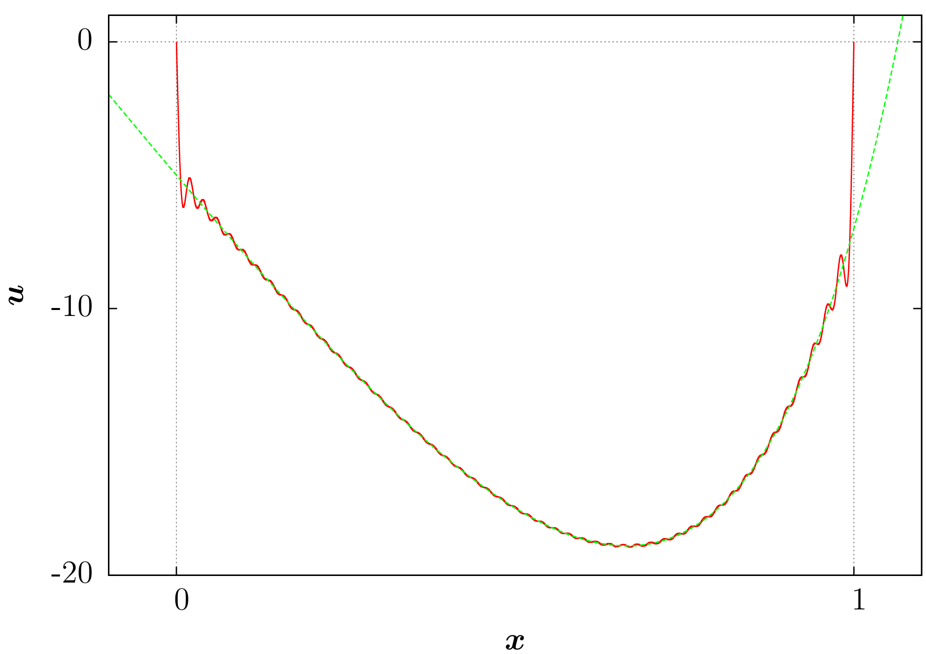
Note that the boundary terms involving $\psi_k u'$ are zero because of the BCs on ψ_k in (4), but the $\psi_k' u$ terms remain because of the BCs on u in (1).[1]

So finally,

$$c_k = \frac{14(-1)^k - 10}{k\pi} + \frac{18(-1+(-1)^k e^4)}{k\pi(16+k^2\pi^2)} \qquad k = 1, 2, 3, \cdots \tag{22}$$

and our solution $u(x)$ is given as the Fourier sine series (15). See the figure for a plot of the Fourier series (first 100 c_k terms) compared against the exact solution of this problem,

$$u_{\text{exact}}(x) = \frac{1}{16}\left(9e^{4x} - [9e^4 + 23]x - 89\right). \tag{23}$$



Note the excellent agreement everywhere except at the boundaries. At $x = 0$ and $x = 1$, we have $\sin(0) = \sin(k\pi) = 0$ so the Fourier series cannot converge to the boundary conditions, but it does the best that it can; observe the Gibbs phenomenon at the jumps where the odd extension cannot match the exact solution, but it converges in L^2 norm.

Also note that none of the steps involved taking derivatives applied to the expansion for $u(x)$. The use of the adjoints and inner products (sometimes called solving the "weak form" of the problem), avoided these issues.[2]

If we had been given the exact solution (23) to start with, we'd get that (22) are its Sine-series coefficients from (15); we have been able to work out the solution by getting the c_k's directly without knowing (23)!

[1] Note that the regular and adjoint eigenfunctions are always defined with homogeneous boundary conditions.

[2] Finite Element Methods (FEM) in numerical analysis also use this general approach to solve problems.