Part 1: Properties of Fourier series expansions

0. For L^2 functions: $f(x) = \sum_{k=1}^{\infty} c_k \phi_k(x) \quad (a.e.) \qquad c_k = \frac{\langle f, \phi_k \rangle}{||\phi_k||^2}$

1. Parseval's theorem: $||f||^2 = \sum_{k=1}^{\infty} c_k^2 ||\phi_k||^2$

2. L^2 convergence of N-term approximation of $f(x)$

$$f(x) = \boxed{\sum_{k=1}^{N} c_k \phi_k(x)} + \sum_{k=N+1}^{\infty} c_k \phi_k(x) \qquad S_N(x) = \sum_{k=1}^{N} c_k \phi_k(x)$$

Show that $S_N(x) \to f(x)$ (a.e.) as $N \to \infty$:

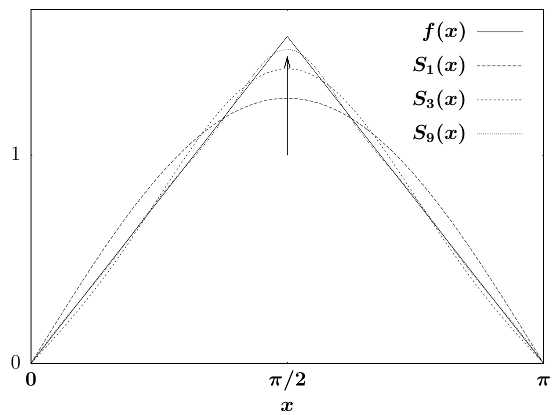
$$\lim_{N\to\infty} ||f(x) - S_N(x)||^2 = \lim_{N\to\infty} \left\| \sum_{k=N+1}^{\infty} c_k \phi_k(x) \right\|^2$$

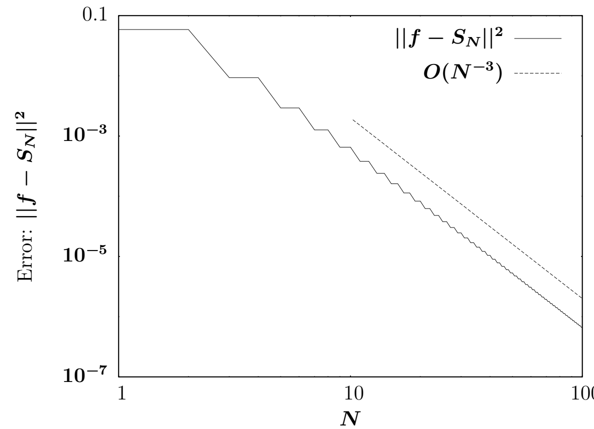
Remainder is the series' "tail". Use orthogonality in $\langle \sum_k c_k \phi_k, \sum_j c_j \phi_j \rangle$

$$= \lim_{N\to\infty} \sum_{k=N+1}^{\infty} c_k^2 ||\phi_k||^2 = 0$$

2. Example: $f(x) = \begin{cases} x & 0 \le x \le \pi/2 \\ \pi - x & \pi/2 < x \le \pi \end{cases}$

   Fourier sine series: $\phi_k(x) = \sin(kx)$ $\qquad c_k = \frac{4}{\pi k^2}\sin(k\pi/2)$





3. <u>Pointwise convergence</u>: If original fcn $f(x)$ is continuous at a pt x_0 then the Fourier series converges there

$$\lim_{N\to\infty} S_N(x_0) = f(x_0)$$

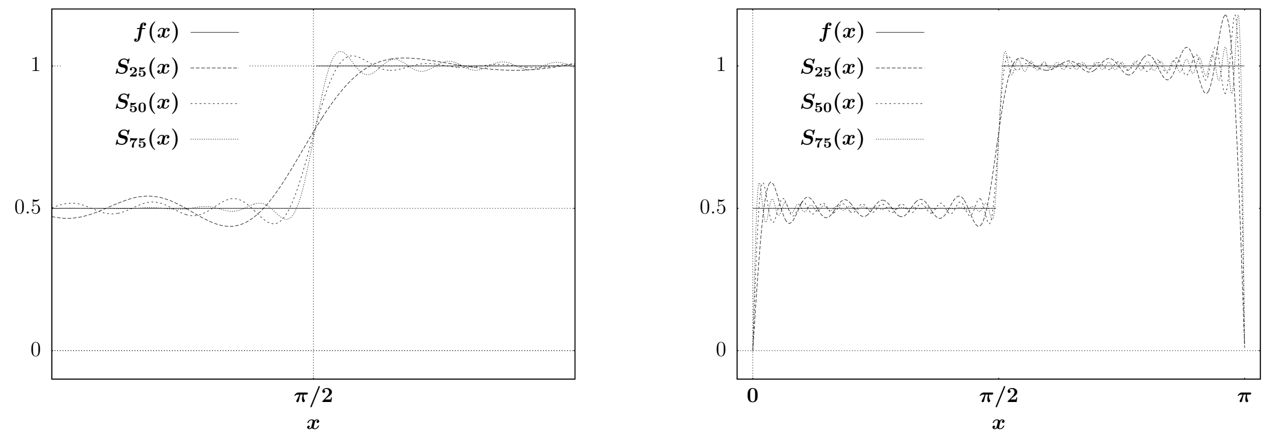
4. If $f(x)$ is <u>discontinuous</u> at a pt x_0 then

$$\lim_{x\to x_0^-} f(x) \neq \lim_{x\to x_0^+} f(x)$$

   then the Fourier series converges to $\boxed{\text{the average of the } f\text{-values at the jump}}$

$$\lim_{N\to\infty} S_N(x_0) = \tfrac{1}{2}[f(x_0^-) + f(x_0^+)]$$

4. Gibbs phenomenon: overshoot/"ringing" of Fourier series approx at a pt x_0 where $f(x)$ has a jump



5. Properties of the c_k coefficients

- If $f(x)$ has jump(s) then its Fourier Series will have coefficients that decay like $\boxed{1/k}$ as $k \longrightarrow \infty$

$$\lim_{k\to\infty} \frac{c_k}{(1/k)} = \text{finite value}$$

- If $f(x)$ is continuous everywhere but it has a corner (so $f'(x)$ has a jump) then $f(x)$'s Fourier Series c_k decay like $\boxed{1/k^2}$ as $k \longrightarrow \infty$.
- If the $f^{(n)}(x)$ derivative is the first to have a jump, then the Fourier series of $f(x)$ have coefficients c_k that decay like $\boxed{1/k^{n+1}}$ as $k \longrightarrow \infty$.

6. Valid operations on Fourier Series: operations on $f(x)$ that translate term-by-term on converging FS's:

- $Af(x) = \sum_k (Ac_k)\phi_k(x)$
- $f(x) + g(x) = \sum_k (c_k + d_k)\phi_k(x)$
- $\int f(x)\,dx = \sum_k c_k \left(\int \phi_k(x)\,dx\right)$

BUT

Differentiation might lead to FS that do not converge (i.e. are wrong!)

Example: $$\frac{d}{dx}\left[x = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2}{k}\sin(kx)\right]$$

So, is $1 = \sum_{k=1}^{\infty} 2(-1)^{k+1}\cos(kx)$?

- NO! Cosine series of $f(x) = 1$ is $f(x) = 1 + \sum_k 0$
- $c_k = \pm 2$ does not have $|c_k| \to 0$ as $k \to \infty$ (n^{th} term test)
- Series DOES NOT converge! (?!)

A big issue for Math 211 is understanding how to work around this problem to obtain guaranteed-correct Fourier Series for solutions of Diff. Eqns.

7. Special properties of Trigonometric Fourier Series:

- $\sin(kx), \cos(kx)$ are 2π-periodic functions, $\phi_k(x+2\pi) = \phi_k(x)$. Most other $\phi_k(x)$ are only defined on interval $a \le x \le b$.
- Functions expanded in these $\phi_k(x)$ "become" periodic extensions of the original $f(x)$ piece on $x \in [-\pi, \pi)$

$$f(x+2\pi) = f(x)$$

- If $f(x)$ is given on $x \in [0, \pi)$ then the Sine Series really gives the expansion of the odd extension of $f(x)$:

$$f_{odd}(x) = \begin{cases} f(x) & x \in [0,\pi) \\ -f(-x) & x \in [-\pi, 0) \end{cases}$$

- If $f(x)$ is given on $x \in [0, \pi)$ then the Cosine Series really gives the expansion of the even extension of $f(x)$:

$$f_{even}(x) = \begin{cases} f(x) & x \in [0,\pi) \\ f(-x) & x \in [-\pi, 0) \end{cases}$$

- Interactions of these extensions (periodic and odd/even) can sometimes produce jumps/Gibbs phenomena at the edges.

$\boxed{\text{Part 2}}$: Extending Lin. Alg. Theory to Lin. Diff. Eqns.

- Solutions/domains

$$\mathbf{u} \in \mathbb{R}^n \qquad \longrightarrow \qquad u(x) \text{ on } a \le x \le b$$

- Inner products

$$\mathbf{u} \cdot \mathbf{v} \qquad \longrightarrow \qquad \int_a^b u(x)v(x)\,dx$$

- Orthogonality

$$\mathbf{u} \cdot \mathbf{v} = 0 \qquad \longrightarrow \qquad \langle u, v \rangle = 0$$

- "Linear operators"

$$\text{matrix } \mathbf{L} \qquad \longrightarrow \qquad \text{"Diff. Op." } L = \text{LHS of ODE}$$

- Problems to be solved

$$\mathbf{L}\mathbf{u} = \mathbf{b} \qquad \longrightarrow \qquad Lu(x) = f(x)$$

Example: LHS is $Lu(x) = p\dfrac{d^2u}{dx^2} + q\dfrac{du}{dx} + ru$

Boundary value problems (BVP) for ODEs have four basic parts:

1. Domain $a \le x \le b$
2. Linear operator L: LHS of eqn $Lu = f$
3. Inhomogeneous forcing function $f(x)$: RHS of eqn $Lu = f$
4. Boundary conditions (BC) at $x = a$ and $x = b$

Without BC's, eqn $Lu = f$ has many possible solutions.
Adding BC's picks one final solution.

- L without BC's = "**Formal Linear Operator**"
- L with BC's = "**Complete Linear Operator**"

Eigenvalue problems

$$\text{Matrix: } \mathbf{L}\phi = \lambda\phi \qquad \text{ODE BVP: } L\phi(x) = -\lambda\phi(x)$$

- Historical tradition: ODE eigenvalues have extra minus sign
- Matrix eigenvalues for $\mathbf{L}_{n\times n}$: determinant $|\mathbf{L} - \lambda\mathbf{I}| = 0$ has n eig-vals
- ODE BVP eigenvalues: has infinite number of eig-vals. (How? IOU)

Adjoint problems: needed to calculate c_k's

- Matrix case: make adjoint $\mathbf{L}^*$ from original $\mathbf{L}$

$$\mathbf{L} \text{ real, then } \mathbf{L}^* = \mathbf{L}^T \qquad \mathbf{L} \text{ complex, then } \mathbf{L}^* = \mathbf{L}^H$$

- ODE BVP: How to make the complete adjoint operator?

$$L^* \text{ from } L$$

$$BC^* \text{ from } BC$$

- The complete adjoint L^* is always defined from the **Inner product relation**

$$\boxed{\langle v(x), Lu(x)\rangle = \langle L^* v(x), u(x)\rangle} \qquad \text{for all } u, v\text{'s}$$

For differential equations, this relation is also called the Lagrange identity or Green's formula.