Math 211, Fall 2009, Duke University Lecture 3

Key elements of Fourier Series and Orthogonal Functions

- Expressing complicated functions as sums of simple functions
- Expressing solutions of differential equations problems as sums
- Reducing DE problems to algebra for c_k coefficients in sums

Inner products for real-valued functions on an interval $a \le x \le b$

$$\boxed{\langle f, g\rangle = \int_a^b [f(x)g(x)]\sigma(x)\,dx}$$

- $\sigma(x) \ge 0$: positive **weight function**
  sometimes called a "weighted inner product"
- For complex-valued functions, the inner product is

$$\langle f, g\rangle = \int_a^b [f(x)\overline{g(x)}]\sigma(x)\,dx$$

- Specifying $\sigma(x)$ and a, b defines an inner product for a problem.

- The uniform weight case: $\sigma(x) \equiv 1$

$$\langle f, g \rangle = \int_a^b [f(x)g(x)]\, dx \qquad \text{"standard } L^2 \text{ inner product"}$$

- L^2 norm

$$(||f||_2)^2 = \langle f, f \rangle = \int_a^b f^2\, dx \geq 0$$

- L^2 functions: aka "square integrable fcns", have finite L^2 norm:

$$||f||_2 < \infty$$

L^2 fcns can blow-up as long they aren't "too badly" behaved:

– Example $f(x) = x^{-1/4}$ on $0 \leq x \leq 1$

$$||f||_2^2 = \int_0^1 (x^{-1/4})^2\, dx = 2x^{1/2}|_0^1 = 2 \qquad \text{Ok, } L^2$$

– Example $f(x) = x^{-1/2}$ on $0 \leq x \leq 1$

$$||f||_2^2 = \int_0^1 (x^{-1/2})^2\, dx = \ln(x)|_0^1 = \infty \qquad \text{NOT } L^2$$

Eigen-expansions are guaranteed to work for L^2 fcns