Math 211, Fall 2009, Duke University — Lecture 2

Review of key parts of Matrix-Vector Eigenvalue Problems (Linear Algebra)

- Let $\mathbf{L}$ be a $n \times n$ square matrix ($\mathbf{L}$ for $\mathbf{L}$inear operator)

- The eigenvalue problem for $\mathbf{L}$:

$$\boxed{\mathbf{L}\phi = \lambda\phi} \qquad |\phi| \neq 0$$

In each of the n "special directions" (eigenvectors ϕ_k) the action of $\mathbf{L}$ is to stretch ϕ_k by factor λ_k (the eigenvalue), without rotating the vector.

- Characteristic polynomial for eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$:

$$\boxed{p(\lambda) = |\mathbf{L} - \lambda\mathbf{I}|} \qquad p(\lambda_k) = 0$$

- **Inner products** are used to define orthogonality, adjoints, norms

- For real vectors, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, use the real inner product:
  $\langle \mathbf{x}, \mathbf{y} \rangle \equiv \mathbf{x} \cdot \mathbf{y}$

- For complex vectors, $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, use the complex inner product:
  $\langle \mathbf{x}, \mathbf{y} \rangle \equiv \mathbf{x} \cdot \bar{\mathbf{y}}$

Important results

- The **adjoint** $\mathbf{L}^*$ is defined by the inner product relation

$$\boxed{\langle \mathbf{y}, \mathbf{L}\mathbf{x} \rangle = \langle \mathbf{L}^* \mathbf{y}, \mathbf{x} \rangle}$$

  - Using the real inner product: $\mathbf{L}^* = \mathbf{L}^T$
  - Using the complex inner product: $\mathbf{L}^* = \bar{\mathbf{L}}^T$

- The adjoint matrix may have different adjoint eigenvectors $\boldsymbol{\psi_k}$, but has the same eigenvalues $\boldsymbol{\lambda_k}$ (conjugated)

$$\boxed{\mathbf{L}^* \boldsymbol{\psi_k} = \overline{\boldsymbol{\lambda_k}} \boldsymbol{\psi_k}}$$

- The **orthogonality** relation for $\boldsymbol{\lambda_j} \neq \boldsymbol{\lambda_k}$.
  From $\boldsymbol{\psi_j} \cdot (\mathbf{L}\boldsymbol{\phi_k} = \boldsymbol{\lambda_k}\boldsymbol{\phi_k}) - \boldsymbol{\phi_k} \cdot (\mathbf{L}^* \boldsymbol{\psi_j} = \overline{\boldsymbol{\lambda_j}} \boldsymbol{\psi_j})$:

$$\boxed{\langle \boldsymbol{\psi_j}, \boldsymbol{\phi_k} \rangle = \mathbf{0}} \qquad \leftrightarrow \qquad \boldsymbol{\psi_j} \perp \boldsymbol{\phi_k}$$

- If $\mathbf{L}$ is **self-adjoint**, $\mathbf{L}^* = \mathbf{L}$, then

$$\boxed{\text{All } \boldsymbol{\lambda_k} \text{ are real}} \qquad \text{and} \qquad \boxed{\text{Each } \boldsymbol{\psi_k} = \boldsymbol{\phi_k}}$$

Completeness: $\{\phi_k\}$ is a complete basis set if every given vector $\mathbf{x} \in \mathbb{R}^n$ can be written as a linear combo:

$$\boxed{\mathbf{x} = \sum_{k=1}^{n} c_k \phi_k} \quad \longrightarrow \quad \text{use } \langle \psi_j, \mathbf{x} \rangle \quad \longrightarrow \quad \boxed{c_j = \frac{\langle \psi_j, \mathbf{x} \rangle}{\langle \psi_j, \phi_j \rangle}}$$

Then, we can use this to solve the linear equation $\boxed{\mathbf{L}\mathbf{x} = \mathbf{b}}$ for unknown $\mathbf{x}$ as an **eigenvector expansion**:

Multiply by each ψ_j for $j = 1, 2, \cdots n$

$$\begin{aligned}
\langle \psi_j, \mathbf{L}\mathbf{x} \rangle &= \langle \psi_j, \mathbf{b} \rangle \\
\langle \mathbf{L}^* \psi_j, \mathbf{x} \rangle &= \\
\langle \overline{\lambda_j} \psi_j, \mathbf{x} \rangle &= \\
\overline{\lambda_j} \langle \psi_j, \mathbf{x} \rangle &= \\
\overline{\lambda_j} c_j \langle \psi_j, \phi_j \rangle &= \langle \psi_j, \mathbf{b} \rangle
\end{aligned}$$

So the formula for the coefficients is

$$\boxed{c_j = \frac{\langle \psi_j, \mathbf{b} \rangle}{\overline{\lambda_j} \langle \psi_j, \phi_j \rangle}}$$

If $\mathbf{L}$ is **self-adjoint**, this reduces to

$$\boxed{c_j = \frac{\langle \phi_j, \mathbf{b} \rangle}{\lambda_j ||\phi_j||^2}}$$