1 Basic elements of Linear Algebra

- Vectors of real numbers $\mathbf{x}, \mathbf{y}, ... \in \mathbb{R}^n$ – column vectors $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

  where all entries x_j are real numbers

- Dot product: product of two vectors, result is a scalar (single number). Standard definition is

$$\mathbf{x} \cdot \mathbf{y} \equiv \sum_{j=1}^{n} x_j y_j = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

  $\mathbf{x} \cdot \mathbf{y}$ can also be computed using the matrix transpose as a row-times-column matrix product:

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = (x_1 \ x_2 \ \cdots \ x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

2 Eigenvalues, eigenvectors of matrices

If $\mathbf{A}$ is a square $n \times n$ real matrix, then the vector $\mathbf{y}$ resulting from the matrix-vector product $\mathbf{y} = \mathbf{A}\mathbf{x}$ generally has the properties:

- $\mathbf{y}$ may have a different direction than $\mathbf{x}$
- $\mathbf{y}$ may have a different magnitude than $\mathbf{x}$

Namely, for general vectors $\mathbf{x}$, the product $\mathbf{A}\mathbf{x}$ is a stretched and rotated vector. For each matrix $\mathbf{A}$ there will be special choices for $\mathbf{x}$ called **eigenvectors** with the resulting $\mathbf{y}$ being in the same direction[1] as $\mathbf{x}$. When $\mathbf{x}$ is along one of the eigenvectors $\boldsymbol{\phi}$, the product $\mathbf{A}\mathbf{x}$ is stretched but *not rotated*. This geometric property will have very important consequences for solving algebra problems.

The length of $\mathbf{y}$ may be different than $\mathbf{x}$, the **eigenvalue** gives the scaling (i.e. stretching or shrinking) of $\mathbf{y}$ relative to $\mathbf{x}$. If the direction of $\mathbf{y}$ is flipped, the eigenvalue will be negative.

If a $n \times n$ matrix has n different eigenvectors then it is called "non-defective" (which means it is the usual, nicer, simpler case) and its set of eigenvectors is said to be **complete**.

Each eigenvalue λ and its associated eigenvector $\boldsymbol{\phi}$ satisfy the equation

$$\mathbf{A}\boldsymbol{\phi} = \lambda\boldsymbol{\phi} \tag{1}$$

where $\boldsymbol{\phi}$ must be a nonzero vector ($|\boldsymbol{\phi}| \neq 0$).

Using the identity matrix $\mathbf{I}$ this equation can be re-written as

$$\mathbf{A}\boldsymbol{\phi} = (\lambda\mathbf{I})\boldsymbol{\phi} \qquad \to \qquad (\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\phi} = \mathbf{0} \tag{2}$$

Recall from linear algebra that if $\mathbf{M}\mathbf{z} = \mathbf{0}$ with $\mathbf{z} \neq \mathbf{0}$ then matrix $\mathbf{M}$ must be singular and its determinant must be zero; this leads to the determinant equation for the eigenvalues of $\mathbf{A}$:

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{3}$$

[1] Actually, $\mathbf{y}$ will be parallel to $\mathbf{x}$, having the same or exactly opposite direction to $\mathbf{x}$.

This is also often called the characteristic polynomial $p(\lambda)$; the zeros of the characteristic potential are the eigenvalues.

For matrices, you start by calculating the eigenvalues from the characteristic polynomial, then you can determine the associated eigenvector for each eigenvalue, one at a time. If λ_k is one of the eigenvalues, write the matrix $\mathbf{A}_k = \mathbf{A} - \lambda_k \mathbf{I}$, then do Gaussian elimination to row echelon form to determine the eigenvector $\boldsymbol{\phi}_k$ of $\mathbf{A}_k \boldsymbol{\phi}_k = \mathbf{0}$.

2.1 Example

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & -4 \\ 1 & -4 & 1 \\ -4 & 0 & 2 \end{pmatrix}$$

Eigenvalues

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 2-\lambda & 0 & -4 \\ 1 & -4-\lambda & 1 \\ -4 & 0 & 2-\lambda \end{vmatrix} = \lambda^3 - 28\lambda - 48 = (\lambda+4)(\lambda-6)(\lambda+2) = 0$$

Eigenvectors

- $\lambda_1 = -4$

  $$\mathbf{A}_1 = \begin{pmatrix} 6 & 0 & -4 \\ 1 & 0 & 1 \\ -4 & 0 & 6 \end{pmatrix}$$

  Equations for $\mathbf{A}_1 \mathbf{x} = \mathbf{0}$

  $$6x_1 - 4x_3 = 0 \qquad x_1 + x_3 = 0 \qquad -4x_1 + 6x_3 = 0$$

  Need $x_1 = 0, x_3 = 0$ but x_2 can be anything. So an eigenvector is

  $$\boldsymbol{\phi}_1 = (0\ 1\ 0)^T$$

- $\lambda_2 = 6$

  $$\mathbf{A}_2 = \begin{pmatrix} -4 & 0 & -4 \\ 1 & -10 & 1 \\ -4 & 0 & -4 \end{pmatrix}$$

  Equations for $\mathbf{A}_2 \mathbf{x} = \mathbf{0}$

  $$-4x_1 - 4x_3 = 0 \qquad x_1 - 10x_2 + x_3 = 0 \qquad -4x_1 - 4x_3 = 0$$

  Need $x_1 = -x_3$ and $x_2 = 0$ but x_3 can be anything. So an eigenvector is

  $$\boldsymbol{\phi}_2 = (1\ 0\ -1)^T$$

- $\lambda_3 = -2$

  $$\mathbf{A}_3 = \begin{pmatrix} 4 & 0 & -4 \\ 1 & -2 & 1 \\ -4 & 0 & 4 \end{pmatrix}$$

  Equations for $\mathbf{A}_3 \mathbf{x} = \mathbf{0}$

  $$4x_1 - 4x_3 = 0 \qquad x_1 - 2x_2 + x_3 = 0 \qquad -4x_1 + 4x_3 = 0$$

  Need $x_1 = x_3$ and $x_2 = x_3$ but x_3 can be anything. So an eigenvector is

  $$\boldsymbol{\phi}_3 = (1\ 1\ 1)^T$$

3 Background

For more review of linear algebra, see the textbook:
Linear algebra and its applications by G. Strang