

## Complex Contour Integrals

- 1. Test 3 (Wed, Dec 7) will cover complex variables and functions (HW 9), and mainly complex integrals (this problem set). (This is the final homework.)
- 0. Reading: See “Methods of contour integration” and related pages on Wikipedia, MathWorld, ...
- 1. (Residues and Heaviside’s short-cut for partial fractions) First, use Cauchy’s integral formula to evaluate the integral on curve  $\mathcal{C}$  going around the three poles in terms of their corresponding residues:

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{dz}{(z - z_a)(z - z_b)(z - z_c)} = A + B + C.$$

What are the formulas for  $A, B, C$ ?

Show that these are the same as the the partial fraction coefficients that you’d get from doing algebra to match coefficients in the LHS vs. RHS below:

$$\frac{1}{(z - z_a)(z - z_b)(z - z_c)} = \frac{A}{z - z_a} + \frac{B}{z - z_b} + \frac{C}{z - z_c}.$$

Namely: combine the RHS fractions over a common denominator and match the RHS numerator polynomial to the LHS polynomial,  $0z^2 + 0z + 1$ , to obtain equations for the coefficients  $A, B, C$ .

(The short-cut via the residue theorem is called “Heaviside’s cover-up method.”)

- 2. On each of the following curves from  $z = 0$  to  $z = 1 + i$  evaluate the integral:

$$\int_{\mathcal{C}} |z|^2 dz$$

- (a) On the diagonal line,  $y = x$ .
- (b) Clockwise on the circle  $|z - 1| = 1$ .
- (c) Counterclockwise on the circle  $|z - i| = 1$ .

- 3. Evaluate on the CCW circle  $C : |z| = \sqrt{2}$

$$\oint_{\mathcal{C}} \frac{\sin(\pi z)}{z^2(z - 1)(z - \frac{1}{2})} dz$$

- 4. (“Loop-de-loop”) Evaluate the integral for the CCW closed contour  $r = 2 - \sin^2(\theta/4)$ ,

$$\oint_{\mathcal{C}} \frac{dz}{z}$$

Hint: Plot the curve, then deform it.

- 5. Evaluate the integral

$$\int_0^\pi \frac{d\theta}{7 + \cos \theta}$$

Hint: The domain is  $[0, \pi]$ , so you’ll first need to extend it to cover a whole circle.

- 6. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{-ix}}{x^3 - i} dx$$

Hint: Be careful with your choice for making a closed contour.

- 7. (“A big slice”)

$$\int_0^\infty \frac{x}{1 + x^3} dx$$

- 8. (“The whole pie”)

$$\int_0^\infty \frac{dx}{x^2 + 3x + 2}$$

- 9. (“Indent this”)

$$\int_0^\infty \left( \frac{\sin x}{x} \right)^2 dx$$

Hint: First use  $\sin^2 x = (1 - \cos(2x))/2$ .

10. (Optional, EXTRA CREDIT) Using contour integration evaluate the integral

$$\int_0^1 \frac{\ln(x)}{x-1} dx$$

Hint: Use  $|z| = 1$  as part of the closed contour and indent around the singularity at  $z = 1$ , and check L'Hopital's rule to see if the singularity is real or "removable" on each side of the branch cut from the log.

11. (Optional, EXTRA CREDIT) The real integral

$$I = \int_0^{2\pi} \frac{d\theta}{5 + 4 \cos \theta}$$

can be evaluated using

$$I = \oint_C f(z) dz \quad \text{where } C \text{ is the unit circle, } |z| = 1.$$

- (a) Use residues to determine the value of the integral.
- (b) Determine Taylor or Laurent series for  $f(z)$  in powers of  $z$  for each of the three rings (annular domains) between this function's two singularities.  
Hint: Use partial fractions to write  $f(z)$  as a sum of two first order poles, then use the geometric series appropriately...
- (c) Show that the series from (b) that is valid on the circle  $|z| = 1$  satisfies  $c_{-n} = \overline{c_n}$  as expected from the Fourier series  $g(\theta) = \sum_n c_n e^{in\theta}$  since  $g(\theta) = 1/(5 + 4 \cos \theta)$  is a real-valued function.
12. (Optional, EXTRA CREDIT) A complex contour integral formula for the Bessel function of order zero is

$$J_0(t) = \frac{1}{2\pi i} \int_C \frac{1}{z} \exp\left(\frac{t}{2} \left[z - \frac{1}{z}\right]\right) dz$$

- (a) Show that if the contour  $C$  is taken as the unit circle,  $z = e^{i\theta}$ , then this can be reduced to

$$J_0(t) = \frac{1}{\pi} \int_0^\pi \cos(t \sin \theta) d\theta$$

- (b) Show that the integral formula from (a) does solve Bessel's equation for  $y(t) = J_0(t)$ :

$$t \frac{d^2 y}{dt^2} + \frac{dy}{dt} + ty = 0$$

Hint: As usual, interchange order of integration and differentiation, and use integration by parts...

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Do not spend too much time on the extra credit problems if you have other work to do, or they seem more trouble than they might be worth...