

Introduction to complex variables and complex functions

0. This problem set will not be collected or graded. A solution set will be provided.
1. Find the roots and state the number of distinct solutions:
- $(z - 4)^3 = -27i$
 - $z^{1.4} = -5$
 - $z^{\sqrt{3}} = 1$
 - $e^z = -2$
 - $z^i = 1$
2. The formulas in terms of exponentials for $\cos x$ and $\sin x$ also apply to $\cos z$ and $\sin z$. Show that:
- $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$ and
 $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$.
 - $\arcsin(z) = -i \ln(iz \pm \sqrt{1 - z^2})$.
 Hint: Let $w = e^{iz}$ and write $\sin(z) = (w - 1/w)/(2i)$ as a quadratic equation in terms of w .
3. If $z = re^{i\theta}$ then define the Principal branch of the logarithm as $\text{Ln}(z) \equiv \ln r + i\theta$ where $0 \leq \theta < 2\pi$. Show that while $(2i) \cdot (-4i) = 8$ it is NOT true that $\text{Ln}(8) = \text{Ln}(2i) + \text{Ln}(-4i)$.
4. Let $f(z) = \ln(1 + \sqrt{z^2 + 1})$.
- What are the branch points of $f(z)$?
 - If $f(\sqrt{3}) = i3\pi$, what is $f(2i) = ?$
5. On connecting real functions $u(x, y)$ to analytic functions $f(z)$: for each $u(x, y)$,

$$u_1(x, y) = e^{-x} \cos(y) \quad u_2(x, y) = \frac{1}{2} \ln(x^2 + y^2)$$

- Show that $u(x, y)$ is a solution of Laplace's equation $\nabla^2 u = 0$.
 - Let $f(z) = u + iv$ where $u = \text{Re}(f(z))$. Use the substitutions $x = (z + \bar{z})/2$, and $y = (z - \bar{z})/(2i)$ in $u(x, y)$ to find $f(z)$ and $v(x, y)$. (Hint: $f(\bar{z}) = \overline{f(z)}$ if $f(x)$ is real.)
6. Derive the Cauchy-Riemann equations in polar coordinates using the following outline of steps:
- Let $f(z) = u(r, \theta) + iv(r, \theta)$ where $z = re^{i\theta}$. We will determine the derivative $f'(z_0)$ at a point, $z_0 = r_0 e^{i\theta_0}$. Use the limit definition of the derivative to find $f'(z_0)$ for two different choices of Δz : first, along a line with $\theta = \theta_0$ fixed, (i.e. $\Delta r \rightarrow 0$), and second, along a curve with $r = r_0$ fixed, (i.e. $\Delta\theta \rightarrow 0$). Hint: Consider how to write Δz from z, z_0 .
 - Equate the real and imaginary parts of the two answers for $f'(z_0)$ to obtain the Cauchy-Riemann equations.
 - Show that cross-differentiating the Cauchy-Riemann equations yields Laplace's equation in polar coordinates.
 - Show that the following u and v satisfy the Cauchy-Riemann equations,

$$u(r, \theta) = e^{-\theta} \cos(\ln r), \quad v(r, \theta) = e^{-\theta} \sin(\ln r)$$

Determine the $f(z)$ that produces this u, v pair.