

Introduction to complex variables and complex functions

1. (Complex algebra) Find the roots and state the number of distinct solutions for each:

$$(a): (z - 5)^3 = -27i \quad (b): z^{1.4} = -8 \quad (c): z^{\sqrt{5}} = 1 \quad (d): e^z = -4 \quad (e): z^i = 3$$

2. (Complex functions) The formulas for $\cos x$ and $\sin x$ in terms of exponentials, $\cos x = (e^{ix} + e^{-ix})/2$ and $\sin x = (e^{ix} - e^{-ix})/(2i)$, also apply to $\cos z$ and $\sin z$. Show that:

(a) $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$

(b) $\arcsin(z) = -i \ln(iz \pm \sqrt{1 - z^2})$.

Hint: Let $w = e^{iz}$ and write $\sin(z) = (w - 1/w)/(2i)$ as a quadratic equation in terms of w .

3. (On connecting real functions $u(x, y)$ to analytic functions $f(z)$, v1.0) Consider $u(x, y) = e^{-x} \cos(y)$:

(a) Show that $u(x, y)$ is a solution of Laplace's equation $\nabla^2 u = 0$.

(b) Let $f(z) = u + iv$ where $u = \operatorname{Re}(f(z))$. Use the substitutions $x = (z + \bar{z})/2$, and $y = (z - \bar{z})/(2i)$ in $u(x, y)$ to find $f(z)$ and $v(x, y)$. (Hint: $f(\bar{z}) = \overline{f(z)}$ if $f(x)$ is real.)

4. (The Cauchy-Riemann equations) Consider $f(z) = u(x, y) + iv(x, y)$ with u, v satisfying the Cauchy-Riemann equations:

(a) Show that $\overline{f(\bar{z})}$ is analytic. (Hint: expand $f(\bar{z})$ in terms of u, v then conjugate that, then apply CR)

(b) Level sets are curves in the xy plane where u or v are constants ($u(x, y) = a$ or $v(x, y) = b$). Show that the levels sets of u and v are orthogonal at any point where they cross.

(c) Show that $u(x, y)$ and $v(x, y)$ each satisfy Laplace's equation, $\nabla^2 u = 0$, $\nabla^2 v = 0$. (Functions that satisfy Laplace's equation are called *harmonic* functions, and the u, v pair are called *harmonic conjugates*.)

(d) Use the second derivative test from multi-variable calculus to show that $u(x, y)$ has no local maxima or minima – show that all critical points must be saddle points.

(e) Use the second derivative test to show that $g(x, y) = |f(z)|^2$ has no local maxima or minima in the complex plane except where $|f| = 0$.

5. (Multi-valued complex functions) See “How to evaluate multi-valued functions” then

(a) If $z = re^{i\theta}$ then define the Principal branch of the logarithm as $\operatorname{Ln}(z) \equiv \ln r + i\theta$ where $0 \leq \theta < 2\pi$. While $(2i) \cdot (-4i) = 8$, show that $\operatorname{Ln}(8) \neq \operatorname{Ln}(2i) + \operatorname{Ln}(-4i)$, find the correct relation.

(b) Let $f(z) = \sqrt{z(1-z)}$. It has branch points at $z_1 = 0$ and $z_2 = 1$.

i. Pick the branch cuts from each to be along the negative horizontal axes, with each θ is in the range $-\pi < \theta \leq \pi$. For each point, determine $r_1, r_2, \theta_1, \theta_2$ and the resulting value of the function: $f(1/2)$, $f(2)$, $f(-1)$ and $\lim_{\epsilon \rightarrow 0} f(1/2 - i\epsilon)$ with $\epsilon \geq 0$.

ii. Pick the same branch cut for z_1 . Pick a branch cut along the positive horizontal axis from z_2 . Determine a range of angles for θ_2 so that $f(1/2) = 1/2$, then determine the value of $\lim_{\epsilon \rightarrow 0} f(2 + i\epsilon)$.