

Eigenfunction expansions for inhomogeneous PDE IBVP (Part 1)

0. Readings from Haberman: Chapter 2 is a very good introduction with easier examples, Chapter 8 goes into more details and harder problems (inhomogeneous BCs and forcing functions)

- Heat equation: Sections 2.3, 2.4.1, Section 8.4.
- Wave equation: Section 8.5 (part 1: pp. 364–368).
- Poisson's/Laplace's equation: Section 2.5.1. Section 8.6 (part 1: pp. 372–275).

1. Consider the problem for the heat equation on $0 \leq x \leq \pi$ and $t \geq 0$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{x^2}{1+t^2}, \quad (1a)$$

$$\left. \frac{\partial u}{\partial x} \right|_{x=0} = \cos(t), \quad \left. \frac{\partial u}{\partial x} \right|_{x=\pi} = e^t, \quad u(t=0) = \sin(x). \quad (1b)$$

Consider the solution of this problem as an eigenfunction expansion in the form $u(x, t) = \sum_k b_k(t)\phi_k(x)$:

- Identify the eigenfunctions $\phi_k(x)$ and eigenvalues λ_k .
- Identify all integrals that need to be evaluated in this problem. Explicitly write them out so they could be evaluated by Maple or Mathematica, but DO NOT work out the integrals.
- Write the ODEs and ICs satisfied by the $b_k(t)$ coefficient functions. Explicitly write out all parts of the problem so they could be evaluated by a computer algebra program, but DO NOT solve the ODE problems.
- This problem has a zero eigenvalue, $\lambda_0 = 0$. Based on the equations that you have constructed for $b_k(t)$, explain if this presents trouble regarding existence or uniqueness of the solution. Calculate the solution $b_0(t)$.

2. Similarly to question 1, consider the problem on $0 \leq x \leq \pi$ for the “damped wave equation”:

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 8t^2, \quad (2a)$$

$$u(x=0) = t^\pi, \quad \left. \frac{\partial u}{\partial x} \right|_{x=\pi} = 3, \quad u(t=0) = 5, \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = \sin(x). \quad (2b)$$

Write down the ODE IVPs for the $b_k(t)$ coefficient functions in the appropriate eigenfunction expansion solution, $u(x, t) = \sum_k b_k(t)\phi_k(x)$. Explain the method you would use to get the solution of the ODE IVPs for the $b_k(t)$ functions.

3. (Does not require solving problems 1 or 2 first) Recall Homework 3, Problem 2 on reducing Gibbs' phenomenon by separating out inhomogeneous boundary conditions.

A similar approach can be applied to many PDE problems. This is done by writing the solution in two parts as $u(x, t) = u_b(x, t) + u_f(x, t)$ where:

- The “boundary solution” u_b satisfies the original inhomogeneous boundary conditions for a homogenized version of the PDE including only the spatial derivative operator, $Lu_b = 0$, and
- The “forced solution” u_f satisfies the full PDE with homogeneous boundary conditions – so u_f can be written as an eigenfunction expansion without Gibbs phenomenon at the boundaries.

The problem for $u_f(x, t)$ will be modified from the original problem depending on the form of $u_b(x, t)$. This is determined by substituting-in $u = u_f + u_b$ into the original problem after you've determined $u_b(x, t)$. (Haberman, Chapter 8.2 is similar).

For problems 1 and 2, the spatial operator is $Lu \equiv \partial^2 u / \partial x^2$.

- For problem 2, determine $u_b(x, t)$, and then determine the PDE problem satisfied by $u_f(x, t)$, namely the PDE with inhomogeneous forcing term, IC's, and BC's. (DO NOT solve this problem.)
- For problem 1, what happens when you try to determine $u_b(x, t)$?
- For problem 1, set $u_b(x, t) = A(t)x + B(t)x^2$. Determine A, B to satisfy the boundary conditions, then determine the PDE problem for $u_f(x, t)$. (DO NOT solve the problem.)

4. A comparison of “x-BVP” vs. “y-BVP” eigenfunction expansions for a Laplace equation problem with Dirichlet and Robin boundary conditions. Consider the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad 0 \leq x \leq \ell \quad 0 \leq y \leq h \quad (3a)$$

$$\text{Left/Right sides:} \quad u(0, y) = 0 \quad u(\ell, y) = 0 \quad 0 \leq y \leq h \quad (3b)$$

$$\text{Bottom/Top sides:} \quad u(x, 0) = 0 \quad u_y(x, h) + 5u(x, h) = f(x) \quad 0 \leq x \leq \ell \quad (3c)$$

- The “x-BVP” eigenfunction expansion takes advantage of the homogeneous left/right BCs. Show that the solution can be written in the form

$$u(x, y) = \sum_{k=1}^{\infty} d_k \sinh\left(\frac{k\pi y}{\ell}\right) \sin\left(\frac{k\pi x}{\ell}\right). \quad (4)$$

and determine the equations for the d_k constants.

- Using the y -direction to determine the set of eigenfunctions yields the solution in the form

$$u(x, y) = \sum_{k=1}^{\infty} b_k(x) \sin(\sqrt{\lambda_k} y). \quad (5)$$

- Homogenize the BCs to determine the equation for the eigenvalues λ_k . (DO NOT try to solve this equation.) Hint: see Chap 5.8.
- Assuming you are given all of the λ_k 's, determine the ODE BVP for the $b_k(x)$ coefficient functions by integrating the projection of problem onto the eigenfunctions (as usual):

$$\int_0^h \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] \phi_k(y) dy = 0 \quad \dots \quad (6)$$