

Eigenfunction expansions for inhomogeneous PDE IBVP (Part 1)

- 1. **Office Hours Shift for Tues Oct 13:** 9:00 am – 11:30 am
(please email for requests for other times on Tues afternoon or Weds morning)
0. Readings from Haberman: Chap 2 is a good intro on easier examples, Chap 8 goes into more details and harder problems (inhomogeneous BCs)
- Heat equation: Sections 2.3, 2.4.1, Section 8.4.
 - Wave equation: Section 8.5 (part 1: pp. 364–368).
 - Poisson's/Laplace's equation: Section 2.5.1. Section 8.6 (part 1: pp. 372–275).

1. Solve the typical looking ODE IVP for coefficient functions $b_k(t)$, $k = 0, 1, 2, \dots$:

$$\frac{db_k}{dt} + kb_k = k(-1)^k t + \int_0^\pi 3 \sin(x-t) \cos(kx) dx \quad b_k(0) = \int_0^\pi 7x \sin(kx) dx \quad (1)$$

Hints: The method of un-determined coefficients is a good approach. Check for any special cases that need to be considered and simplify for even/odd k .

2. Consider the problem for the heat equation on $0 \leq x \leq \pi$ and $t \geq 0$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + xt, \quad (2a)$$

$$\frac{\partial u}{\partial x} \Big|_{x=0} = \cos(t), \quad \frac{\partial u}{\partial x} \Big|_{x=\pi} = e^{-t}, \quad u(t=0) = \sin(x). \quad (2b)$$

Write your solution as an eigenfunction expansion in the form $u(x, t) = \sum_k b_k(t) \phi_k(x)$:

- (a) Identify the eigenfunctions $\phi_k(x)$.
 - (b) Identify all integrals that need to be evaluated in this problem.
DO NOT work out the integrals.
 - (c) Write the ODEs and ICs satisfied by the $b_k(t)$ coefficient functions.
DO NOT solve the ODE problems.
 - (d) This problem has a zero eigenvalue, $\lambda_0 = 0$. Based on the equations that you have constructed for $b_k(t)$, explain if this presents trouble regarding existence or uniqueness of the solution.
3. Similarly to question 2, consider the problem on $0 \leq x \leq \pi$ for the “damped wave equation”:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + t, \quad (3a)$$

$$u(x=0) = \arctan(t), \quad \frac{\partial u}{\partial x} \Big|_{x=\pi} = 1, \quad u(t=0) = 3, \quad \frac{\partial u}{\partial t} \Big|_{t=0} = 5 \cos(x). \quad (3b)$$

Write down the ODE IVPs for the $b_k(t)$ coefficient functions in the appropriate eigenfunction expansion solution. Explain the method you would use to get the solution of the ODE IVPs for the $b_k(t)$ functions.

(Identify but DO NOT work out any integrals needed.)

(continued)

4. A comparison of “ x -BVP” vs. “ y -BVP” eigenfunction expansions for a Laplace equation problem with Robin boundary conditions. Consider the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad 0 \leq x \leq \ell \quad 0 \leq y \leq h \quad (4a)$$

$$\text{Left/Right sides:} \quad u(0, y) = 0 \quad u(\ell, y) = 0 \quad 0 \leq y \leq h \quad (4b)$$

$$\text{Bottom/Top sides:} \quad u(x, 0) = 0 \quad u_y(x, h) - 2u(x, h) = f(x) \quad 0 \leq x \leq \ell \quad (4c)$$

(a) The “ x -BVP” eigenfunction expansion takes advantage of the homogeneous left/right BCs. Show that the solution can be written in the form

$$u(x, y) = \sum_{k=1}^{\infty} d_k \sinh\left(\frac{k\pi y}{\ell}\right) \sin\left(\frac{k\pi x}{\ell}\right). \quad (5)$$

and determine the equations for the d_k constants.

(b) Using the y -direction to determine the set of eigenfunctions yields the solution in the form

$$u(x, y) = \sum_{k=1}^{\infty} b_k(x) \sin(\sqrt{\lambda_k} y). \quad (6)$$

i. Homogenize the BCs to determine the equation for the eigenvalues λ_k .
(DO NOT try to solve this equation.) Hint: see Chap 5.8.

ii. Assuming you are given all of the λ_k 's, determine the ODE BVP for the $b_k(x)$ coefficient functions by integrating the projection of problem onto the eigenfunctions (as usual):

$$\int_0^h \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] \phi_k(y) dy = 0 \quad (7)$$
