

Green's Functions for ODE BVP

0. Read Haberman, section 9.3 (pages 385–399).
1. Haberman, page 400. Problem 9.3.6(a) and plot $g(x, s)$ vs. x .
2. (Verifying Mercer's Theorem) Consider the inhomogeneous boundary value problem on $0 \leq x \leq \pi/2$,

$$\frac{d^2 u}{dx^2} + u = f(x) \quad u(0) = 0 \quad u\left(\frac{\pi}{2}\right) = 0. \quad (1)$$

- (a) Work-out the variation-of-parameters form of the solution.
- (b) Use the variation-of-parameters solutions to write the piecewise-defined Green's function for

$$u(x) = \int_0^{\pi/2} g(x, s) f(s) ds. \quad (2)$$

- (c) Find the eigenfunctions and eigenvalues for

$$\frac{d^2 \phi}{dx^2} + \phi = -\lambda \phi, \quad \phi(0) = 0, \quad \phi\left(\frac{\pi}{2}\right) = 0. \quad (3)$$

- (d) Calculate the coefficients in the eigenfunction expansion of $g(x, s)$ [the piecewise-defined function used in (2)],

$$g(x, s) = \sum_{k=1}^{\infty} g_k \phi_k(x) \quad (4)$$

Hint: Think of g just as a function of x (with s as a parameter).

- (e) Write the solution of problem (1) in terms of an expansion using the eigenfunctions,

$$u(x) = \sum_{k=1}^{\infty} c_k \phi_k(x), \quad (5)$$

and note the relation between g_k and c_k .

3. Haberman, page 403. Problem 9.3.23.
4. In lecture, we determined the Green's function satisfying

$$\frac{d^2 G}{dx^2} + G = \delta(x - s), \quad G(x = 0) = 0, \quad G(x = 1) = 0$$

is

$$G(x, s) = \begin{cases} \sin(s-1)\sin(x)/\sin(1) & 0 \leq x < s \\ \sin(s)\sin(x-1)/\sin(1) & s < x \leq 1 \end{cases}$$

Use this Green's function to determine A, B, C in the solution $u(x) = A \cos(x) + B \sin(x) + Ce^x$ of the inhomogeneous problem

$$\frac{d^2 u}{dx^2} + u = 2e^x \quad u(0) = 5, \quad u(1) = -2$$

See Haberman, section 9.3.5, pages 397–399.