

Singular Sturm-Liouville, Fredholm's Alternative, and Integral Equations

1. Legendre's equation: Consider bounded solutions of the Sturm-Liouville eigenvalue problem on the interval $-1 \leq x \leq 1$ for

$$\frac{d}{dx} \left[(1-x^2) \frac{d\phi}{dx} \right] + \lambda\phi = 0. \tag{1}$$

- (a) What are $p(x), q(x), \sigma(x)$? Explain why no boundary conditions need to be specified.
 - (b) Writing (1) as $L\phi + \lambda\sigma\phi = 0$, use the inner product relation $\langle v, Lu \rangle$ to directly compute L^* and show that the boundary terms will vanish for any smooth, bounded functions u, v .
 - (c) What orthogonality relation do the eigenfunctions satisfy?
 - (d) Let $\phi_0(x) \equiv 1$. Observe that this is an eigenfunction of $L\phi_0 = -\lambda_0\sigma\phi_0$ with $\lambda_0 = 0$. Let $w_k(x) \equiv x^k$ for $k = 1, 2, 3, \dots$. Use the Gram-Schmidt orthogonalization process¹ with the orthogonality relation from (c) to construct the eigenfunctions ϕ_1, ϕ_2, ϕ_3 from w_1, w_2, w_3 .²
 - (e) Evaluate $L\phi_k$ for $k = 1, 2, 3$ to determine the corresponding eigenvalues λ_k .
 - (f) Verify by direct substitution that $u(x) = A \ln \left(\frac{1+x}{1-x} \right) + B$ is a solution of $Lu = 0$ for any values of A and B . Is the 'A' term bounded on the whole domain?³ What value for A makes $u(x)$ a bounded solution for this problem?
 - (g) Find the general solution of $Lu = 1$ by direct integration. Is there an "acceptable" (bounded) solution for this problem?
 - (h) In order for the inhomogeneous problem $Lu = f(x)$ to have a solution in the form of an eigenfunction expansion, $u(x) = \sum_k c_k \phi_k(x)$, what solvability condition must $f(x)$ satisfy?
2. Use the Fredholm alternative theorem to determine the parameter values (A, B) that yield existence of a solution for each inhomogeneous boundary value problem: (Show all work! Do NOT determine the complete solution $u(x)$!)

- (a) For $0 \leq x \leq 2\pi$:

$$\frac{d^2u}{dx^2} + u = A \sin x + B \cos x + 5\pi e^{2x} \quad u(0) = u(2\pi) \quad u'(0) = u'(2\pi).$$

Hint: The hom. problem is self-adjoint and has two lin. inde. adjoint solutions for $\lambda = 0$.

- (b) For $0 \leq x \leq \pi/2$:

$$\frac{d^2u}{dx^2} + 6 \frac{du}{dx} + 9u = 6 \quad u'(0) + 3u(0) = 7 \quad u'(\frac{\pi}{2}) + 3u(\frac{\pi}{2}) = A.$$

Hint: Recall HW3, Q3(b) – what is the adjoint eigenfunction for $\lambda = 0$?

- (c) For $0 \leq x \leq \pi/2$: Find the relation between A and B

$$\frac{d^2u}{dx^2} + 6 \frac{du}{dx} + 9u = 6 + Ae^{-3x} \quad u'(0) + 3u(0) = B \quad u'(\frac{\pi}{2}) + 3u(\frac{\pi}{2}) = 0.$$

¹Recall from linear algebra: To construct an orthogonal set of vectors $\{\vec{v}_k\}$ from an ordered set of linearly independent vectors $\{\vec{u}_k\}$, subtract-off from \vec{u}_k all of the projections of \vec{u}_k onto the previously generated \vec{v}_j ($j = 0, \dots, k-1$) vectors:

$$\begin{aligned} \vec{v}_0 &= \vec{u}_0 \\ \vec{v}_1 &= \vec{u}_1 - \frac{\langle \vec{u}_1, \vec{v}_0 \rangle}{\langle \vec{v}_0, \vec{v}_0 \rangle} \vec{v}_0 \\ \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_0 \rangle}{\langle \vec{v}_0, \vec{v}_0 \rangle} \vec{v}_0 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 \\ \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_0 \rangle}{\langle \vec{v}_0, \vec{v}_0 \rangle} \vec{v}_0 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2 \end{aligned}$$

and so on. Observe that $\langle \vec{v}_k, \vec{v}_j \rangle = 0$ for any $k \neq j$.

²The eigenfunctions are called the *Legendre polynomials*, $P_k(x)$ (also called the Legendre functions of first kind).

³This term is can be written as $AQ_0(x)$ where $Q_0(x)$ is called one of the Legendre functions of second kind, $Q_k(x)$.

3. Solution of FIE's (v1.0): Solutions of first kind Fredholm integral equations can be obtained by direct substitution of $u(x) = \sum_{j=1}^n d_j \alpha_j(x)$ into the equation. Find a solution or show that no solution exists for

(a) $\int_0^1 (x - 2x^2t)u(t) dt = x - 3x^2$

(b) $\int_0^1 \sin(2\pi x - \pi t)u(t) dt = \sqrt{2} \sin(2\pi x + \frac{\pi}{4})$

(c) $\int_0^1 \sin(2\pi x - \pi t)u(t) dt = 3 \cos(\pi x) - 2 \sin(2\pi x)$

(d) $\int_0^\infty [e^{-2x-3t} - 6e^{-x-4t}] u(t) dt = 4e^{-x}$

4. Solution of FIE's (v2.0)

(a) Re-examine the integral operator $Lu \equiv \int_0^1 (x - 2x^2t)u(t) dt$:

- i. Find the eigenvalues of finite multiplicity and their eigenfunctions for $L\phi_k = \lambda_k\phi_k$.
- ii. Write the adjoint operator and determine the adjoint eigenfunctions for the eigenvalues of finite multiplicity for $L^*\psi_k = \lambda_k\psi_k$.
- iii. Fredholm integral operators of first kind have zero as an eigenvalue of infinite multiplicity, $\lambda^\infty = 0$. Demonstrate this by constructing a set of adjoint eigenfunctions satisfying $L^*\psi_m^\infty = 0$ using the guess $\psi_m^\infty(x) = c_1 + mx + c_2x^m$ for $m = 2, 3, \dots$. Find the $c_1(m), c_2(m)$'s. Explain why $m = 1$ is not useful.
- iv. Show orthogonality: that $\phi_1 \perp \psi_m^\infty$ and $\phi_2 \perp \psi_m^\infty$.

(b) Consider the Fredholm integral operator of second kind $Lu \equiv 4u(x) + \int_0^1 \sin(2\pi x - \pi t)u(t) dt$:

- i. Find the eigenvalues of finite multiplicity and their eigenfunctions for $L\phi_k = \lambda_k\phi_k$.
- ii. Show that $\lambda^\infty = 4$ is of infinite multiplicity by showing that there is an infinite set of sine's and cosine's that satisfy $L\phi = 4\phi$.
- iii. What are the adjoint eigenfunctions $\psi_m^\infty(x)$ for λ^∞ ?
- iv. Find the unique solution of $Lu = 8x$ by direct substitution of $u(x) = px + \sum_{j=1}^n d_j \alpha_j(x)$ into Lu to determine unknown coefficients p, d_j .

Math 211 Test 1: Lectures 3–10, Homeworks 2–4, Solution of inhomogeneous ODE BVP via eigenfunction expansions. Fourier series. Adjoint eigenvalue problems. Sturm-Liouville problems. Integral equations: eigenvalue and inhomogeneous problems. The Fredholm alternative theorem for existence/uniqueness/non-existence.

No books, no calculators. A copy of the 'basic mathematics summary' will be provided and you may bring one letter-sized sheet (2 sides) of handwritten notes.