

Singular Sturm-Liouville, Fredholm's Alternative, and Integral Equations

1. Legendre's equation: Consider bounded solutions of the Sturm-Liouville eigenvalue problem on the interval $-1 \leq x \leq 1$ for

$$\frac{d}{dx} \left[(1-x^2) \frac{d\phi}{dx} \right] + \lambda\phi = 0. \quad (1)$$

- (a) What are $p(x), q(x), \sigma(x)$? Why are no boundary conditions specified?
- (b) Writing (1) as $L\phi + \lambda\sigma\phi = 0$, use the inner product relation $\langle v, Lu \rangle$ to directly compute L^* and show that the boundary terms will vanish for any smooth, bounded functions u, v .
- (c) What orthogonality relation do the eigenfunctions satisfy?
- (d) Let $\phi_0(x) = 1$. Observe that this is an eigenfunction of $L\phi_0 = -\lambda_0\sigma\phi_0$ with $\lambda_0 = 0$. Let $w_k(x) = x^k$ for $k = 1, 2, 3, \dots$. Use the Gram-Schmidt orthogonalization process¹ with the orthogonality relation from (c) to construct the eigenfunctions ϕ_1, ϕ_2, ϕ_3 from w_1, w_2, w_3 .²
- (e) Evaluate $L\phi_k$ for $k = 1, 2, 3$ to determine the corresponding eigenvalues λ_k .
- (f) Verify by direct substitution that $u(x) = A \ln \left(\frac{1+x}{1-x} \right) + B$ is a solution of $Lu = 0$ for any values of A and B . Is the 'A' term bounded on the whole domain?³ What value for A makes $u(x)$ a bounded solution for this problem?
- (g) Find the general solution of $Lu = 1$ by direct integration. Is there an "acceptable" (bounded) solution for this problem?
- (h) In order for the inhomogeneous problem $Lu = f(x)$ to have a solution in the form of an eigenfunction expansion, $u(x) = \sum_k c_k \phi_k(x)$, what solvability condition must $f(x)$ satisfy?
2. Use the Fredholm alternative theorem to determine the parameter values (A, B) that yield existence of a solution for each inhomogeneous boundary value problem:

- (a) For $0 \leq x \leq 2\pi$:

$$\frac{d^2u}{dx^2} + u = A \sin x + B \cos x + 2 \sin(x + \frac{\pi}{3}) + \sin^3 x \quad u(0) = u(2\pi) \quad u'(0) = u'(2\pi).$$

Hint: The homogeneous problem is self-adjoint and that there are two linearly independent adjoint solutions for $\lambda = 0$.

- (b) For $0 \leq x \leq 1$:

$$\frac{d^2u}{dx^2} + 2\frac{du}{dx} + u = 1 \quad u'(0) + u(0) = A \quad u'(1) + u(1) = 3.$$

Hint: Recall HW3, Q3(b) – what is the adjoint eigenfunction for $\lambda = 0$?

- (c) For $0 \leq x \leq 1$:

$$\frac{d^2u}{dx^2} + 2\frac{du}{dx} + u = Ae^{-x} \quad u'(0) + u(0) = 5 \quad u'(1) + u(1) = \pi/e.$$

¹See the Lecture 10 handout.

²The eigenfunctions are called the *Legendre polynomials*, $P_k(x)$ (also called the Legendre functions of first kind).

³This term can be written as $AQ_0(x)$ where $Q_0(x)$ is called one of the Legendre functions of second kind, $Q_k(x)$.

3. Solution of FIE's (v1.0): Solutions of first kind Fredholm integral equations can be obtained by direct substitution of $u(x) = \sum_{k=1}^n d_k \alpha_k(x)$ into the equation. Find a solution or show that no solution exists for

(a) $\int_0^1 (x - 2x^2t)u(t) dt = 3x - 5x^2$

(b) $\int_0^1 \sin(2\pi x - \pi t)u(t) dt = 4\sqrt{2} \cos(2\pi x - \frac{\pi}{4})$

(c) $\int_0^1 \sin(2\pi x - \pi t)u(t) dt = \sin(\pi x)$

(d) $\int_0^\infty [20e^{-x-3t} - 36e^{-2x-2t}] u(t) dt = e^{-x}$

4. Solution of FIE's (v2.0)

(a) Re-examine the integral operator $Lu \equiv \int_0^1 (x - 2x^2t)u(t) dt$:

- i. Find the eigenvalues of finite multiplicity and their eigenfunctions for $L\phi_k = \lambda_k\phi_k$.
- ii. Write the adjoint operator and determine the adjoint eigenfunctions for the eigenvalues of finite multiplicity for $L^*\psi_k = \lambda_k\psi_k$.
- iii. Fredholm integral operators of first kind have zero eigenvalues of infinite multiplicity, $\lambda^\infty = 0$. Demonstrate this by constructing a set of adjoint eigenfunctions satisfying $L^*\psi_{\infty,m} = 0$ using the guess $\psi_{\infty,m}(x) = 1 + a_1x + a_2x^m$ for $m = 1, 2, 3, \dots$. Find the $a_1(m), a_2(m)$'s.
- iv. Show that all the ϕ, ψ satisfy orthogonality.

(b) Consider the Fredholm integral operator of second kind $Lu \equiv 2u(x) + \int_0^1 \sin(2\pi x - \pi t)u(t) dt$:

- i. Find the eigenvalues of finite multiplicity and their eigenfunctions for $L\phi_k = \lambda_k\phi_k$.
- ii. Show that $\lambda^\infty = 2$ is of infinite multiplicity by showing that there is an infinite set of sine's and cosine's that satisfy $L\phi = 2\phi$.
- iii. What are the adjoint eigenfunctions $\psi_{\infty,m}(x)$ for λ^∞ ?
- iv. Find the solution of $Lu = x$ by direct substitution of $u(x) = px + \sum_{k=1}^n b_k \alpha_k(x)$ into Lu .
- v. Find the coefficients in Fourier eigenfunction expansion of the 'px' term from (iv) using the $\phi_\infty(x)$ eigenfunctions from (ii).

Math 211 Test 1: Solution of inhomogeneous ODE BVP via eigenfunction expansions. Fourier series. Adjoint eigenvalue problems. Sturm-Liouville problems. Integral equations: eigenvalue and inhomogeneous problems. The Fredholm alternative theorem for existence/uniqueness/non-existence.

Lectures 3–10, Homeworks 2–4.

No books, no calculators. A copy of the 'basic mathematics summary' will be provided and you may bring one sheet (2 sides) of handwritten notes.