

Eigenvalue and Boundary Value Problems for Linear ODEs

0. Reading: Haberman, Sections 5.3, 5.5, 5.10.
1. Sturm-Liouville form for all second order equations – Consider the classic eigenvalue problem $L\phi = -\lambda\phi$ for general second order linear operators

$$A(x)\frac{d^2\phi}{dx^2} + B(x)\frac{d\phi}{dx} + C(x)\phi = -\lambda\phi \tag{1}$$

where $A(x), B(x), C(x)$ are given functions.¹ Show that (1) can always be put into Sturm-Liouville form,

$$\frac{d}{dx} \left(p(x)\frac{d\phi}{dx} \right) + q(x)\phi = -\lambda\sigma(x)\phi. \tag{2}$$

Do this by determining $p(x), q(x), \sigma(x)$ in terms of $A(x), B(x), C(x)$ through the steps:

- Expand the product rule term in (2).
 - Multiply (2) across by an unknown integrating factor function, $m(x)$, and set that equation equal to (1) to obtain four coefficient equations for p, q, σ, m .
 - First, solve the ODE in terms of m alone (i.e. no p, q, σ) to obtain $m(x)$ and then get $p(x), q(x), \sigma(x)$.
2. Reducing Gibbs' phenomenon – Sometimes it can be important to obtain formulas for the solution of a BVP that minimize Gibbs' phenomenon issues at the boundaries. This question illustrates how to do that *for a small price* – that the solution will be written in two pieces: an eigenfunction expansion and a particular solution to satisfy the inhomogeneous boundary conditions. The problem

$$Lu(x) = f(x) \quad BC_1(a) = c \quad BC_2(b) = d \tag{3}$$

can be solved by writing the solution as $u(x) = u_b(x) + w(x)$ where

- $u_b(x)$ satisfies the unforced equation (RHS= 0) but with the inhomogeneous boundary conditions

$$Lu_b(x) = 0 \quad BC_1(a) = c \quad BC_2(b) = d \tag{4}$$

- Once $u_b(x)$ has been determined, then plug-in $u(x) = u_b + w$ into the full problem to write a reduced problem for $w(x)$

$$Lw(x) = f(x) \quad BC_1(a) = 0 \quad BC_2(b) = 0 \tag{5}$$

Re-Solve the example BVP from Lecture 6: $\frac{d^2u}{dx^2} = 9e^{4x}, \quad u(0) = -5, \quad u(1) = -7$ in two pieces:

- (a) Find $u_b(x)$ as a polynomial function that solves $u_b'' = 0, \quad u_b(0) = -5, \quad u_b(1) = -7$.
- (b) Find $w(x)$ as an eigenfunction expansion.
- (c) Show that your solution matches the coefficients c_k , equation (22), found for $u(x)$ in Lecture 6.
- (d) The series for $w(x)$ still has some Gibbs' phenomena in it, but at a "weaker level". Based on its series coefficients for $k \rightarrow \infty$, how smooth is $w(x)$? (what is the lowest derivative of $w(x)$ that is discontinuous?)

¹Haberman's question 5.3.3 is similar. This is also the ODE analogue of the weighted inner product for the non-symmetric matrix, question 5.5A.3, Homework # 1 question 5.

3. Consider the eigenvalue problem on $0 \leq x \leq 1$,

$$\frac{d^2\phi}{dx^2} + 2\frac{d\phi}{dx} + (1 + \lambda)\phi = 0,$$

$$\phi'(0) + \phi(0) = 0 \quad \phi'(1) + \phi(1) = 0.$$

- Assuming λ to be a positive constant, what is the general solution of the homogeneous ODE. Apply the boundary conditions to determine the eigenvalues and eigenfunctions.
- What is the adjoint problem? Obtain the adjoint eigenfunctions.
- What are the functions p, q, σ that put this problem in standard Sturm-Liouville form? (see Problem 1)
- Using (a) and (b) write the simplest forms for the integrals for the “old” ($\langle \phi_k, \psi_j \rangle_{old}$) and “new” ($\langle \phi_k, \phi_j \rangle_{new}$) orthogonality conditions and show that they are equivalent to each other.
- Obtain the coefficients c_k in the expansion of the function

$$u(x) = \sum_{k=0}^{\infty} c_k \phi_k(x),$$

that solves the boundary value problem,

$$\frac{d^2u}{dx^2} + 2\frac{du}{dx} + 2u = 1,$$

$$u'(0) + u(0) = 2 \quad u'(1) + u(1) = 3.$$

Hint: If $\phi(x)$ is a solution of $L\phi = -\lambda\phi$ then it is also a solution of $L\phi - c\phi = -\tilde{\lambda}\phi$ where $\tilde{\lambda} = \lambda + c$.

4. Another inhomogeneous boundary value problem:

- Find the general homogeneous solution of the Cauchy-Euler equation

$$x^2 \frac{d^2\phi}{dx^2} + 3x \frac{d\phi}{dx} + (1 + \alpha)\phi = 0,$$

where α is a given positive constant.

- Use (a) to determine the eigenvalues and eigenfunctions of the Sturm-Liouville problem on $1 \leq x \leq e$

$$\frac{d}{dx} \left(x^3 \frac{d\phi}{dx} \right) + \lambda x \phi = 0 \quad \phi(1) = 0 \quad \phi(e) = 0.$$

- Use (b) to obtain the eigenfunction expansion for the solution of the inhomogeneous problem

$$\frac{d}{dx} \left(x^3 \frac{du}{dx} \right) = x \quad u(1) = 0 \quad u(e) = 0.$$