

Linear Algebra Review

- 1. Office hours: Unless I receive requests to shift the office hours for the course to a different time (due to people's schedule conflicts), usual hours will be Tuesdays, 10:00am-12:30. This upcoming Tuesday (Sep 1), I am on a Ph.D. defense committee in the morning, so on Tues Sep 1, office hours will be 2:00-5:00 pm. You may always email me with your questions or to request to schedule a time to talk with me.
0. Reading: Haberman, Appendix to 5.5 (pp. 184–188).
1. Using the following steps solve the linear system in terms of an eigenvector expansion:

$$\begin{pmatrix} 2 & 0 & -4 \\ 1 & -4 & 1 \\ -4 & 0 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 36 \\ 24 \end{pmatrix}.$$

You are given that the eigenvalues and eigenvectors for $\mathbf{L}\phi = \lambda\phi$ are

$$\{\lambda_1 = -4, \quad \phi_1 = (0, 1, 0)^T\} \quad \{\lambda_2 = 6, \quad \phi_2 = (1, 0, -1)^T\} \quad \{\lambda_3 = -2, \quad \phi_3 = (1, 1, 1)^T\}$$

- (a) Show that $\{\phi_1, \phi_2, \phi_3\}$ are not orthogonal, but they are linearly independent.
Hint: What is the method to check for linear independence via a determinant calculation?
- (b) Find the adjoint eigenvectors $\{\psi_1, \psi_2, \psi_3\}$. Let the first entry of each ψ_k be 1.
- (c) Determine the expansion coefficients c_k and compute the solution $\mathbf{x} = \sum c_k \phi_k$.
2. For vectors whose entries are complex numbers the inner product is defined as $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \bar{\mathbf{y}}$, where $\bar{\mathbf{y}}$ is the complex conjugate, entry by entry, of \mathbf{y} . ($\overline{a + ib} = a - ib$)
- (a) Show that the 'real inner product' is not a norm for complex vectors.
Hint: Use the counterexample $\mathbf{x} = (1, i)^T$, explain.
- (b) Let $\mathbf{x} = (a + ib, c + id)^T$, where a, b, c, d are real numbers. Show that the 'complex inner product' is a norm, with $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle \geq 0$.
- (c) How is the value of the complex inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ related to the value of $\langle \mathbf{y}, \mathbf{x} \rangle$?
3. (a) Problem 5.5A.6, page 189.
(b) Problem 5.5A.5, part (b), page 189.
(c) Find $\{\lambda_k, \phi_k, \psi_k\}$ for $\mathbf{A} = \begin{pmatrix} i & -1 \\ 2 & i + 2 \end{pmatrix}$ and show that $\phi_1 \perp \psi_2$ and $\phi_2 \perp \psi_1$.
4. Problem 5.5A.4, part (a), page 189.
5. (a) Problem 5.5A.3, page 189.
(b) Find the adjoint eigenvectors of \mathbf{A} . Verify the orthogonality relation for $\phi_i \cdot \psi_j$.
(c) Let $\mathbf{M} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}$. Show that the book's " $\mathbf{a} \cdot \mathbf{b}$ " dot product can be written in terms of the usual dot product as $\mathbf{a} \cdot \mathbf{M}\mathbf{b}$.
(d) Find the positive diagonal matrix \mathbf{C} such that $\mathbf{M} = \mathbf{C}^2$.
(e) Multiply the eigenvalue equation $\mathbf{A}\phi = \lambda\phi$ on the left by \mathbf{M} to get $\mathbf{M}\mathbf{A}\phi = \lambda\mathbf{M}\phi$. Write $\mathbf{M} = \mathbf{C}^2$ and $\phi = \mathbf{C}^{-1}\mathbf{y}$ in this equation. Show that \mathbf{y} satisfies $\mathbf{B}\mathbf{y} = \lambda\mathbf{y}$, where \mathbf{B} is a symmetric matrix – Find \mathbf{B} .
(f) Explain why the orthogonality relation for the \mathbf{y} 's is $\mathbf{y}_1 \cdot \mathbf{y}_2 = 0$ and use it to justify the orthogonality of the ϕ 's in the " $\mathbf{a} \cdot \mathbf{b}$ " dot product from (c). Hint: use $\mathbf{y} = \mathbf{C}\phi$.