

Linear Algebra Review

- 1. Homework policy (reminder): Homework is due IN CLASS on the due date. Unexcused late homeworks will not be accepted. Any extensions or excuses must be requested before the due date.

Office hours: We will vote on regular office hour times, but for next week, Tues, Sep 6, office hours will be 12:00-3:00pm. You may always email me with your questions or to request to schedule a time to talk with me.

0. Reading: Haberman, Appendix to section 5.5 (pp. 184–188).
1. (The eigen-expansion method for solving systems of linear equations)  
Consider the matrix equation  $\mathbf{L}\mathbf{x} = \mathbf{b}$ :

$$\begin{pmatrix} -8 & 14 & -14 \\ -5 & 7 & -5 \\ -5 & 1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 20 \\ -3 \\ -15 \end{pmatrix}.$$

You are given that the eigenvalues and eigenvectors for  $\mathbf{L}\phi = \lambda\phi$  are

$$\{\lambda_1 = -8, \quad \phi_1 = (2, 1, 1)^T\} \quad \{\lambda_2 = 2, \quad \phi_2 = (0, 1, 1)^T\} \quad \{\lambda_3 = 6, \quad \phi_3 = (-1, 0, 1)^T\}$$

- (a) Show that  $\{\phi_1, \phi_2, \phi_3\}$  are not orthogonal, but they are linearly independent.  
Hint: First test all pairs of eigenvectors for orthogonality, then look up linear algebra books for tests for linear independence of a set of vectors (one way uses a determinant calculation).
- (b) Find the adjoint eigenvectors  $\{\psi_1, \psi_2, \psi_3\}$ . Set the last entry of each  $\psi_k$  be 1.
- (c) Determine the expansion coefficients  $c_k$  and compute the solution  $\mathbf{x} = \sum c_k \phi_k$ .
2. (Linear algebra with complex vectors)  
For vectors whose entries are complex numbers the inner product is defined as  $\langle \mathbf{x}, \mathbf{y} \rangle \equiv \mathbf{x} \cdot \bar{\mathbf{y}}$ , where  $\bar{\mathbf{y}}$  is the complex conjugate of vector  $\mathbf{y}$ , entry by entry in the vector, with  $a + ib = a - ib$ .
- (a) Show that the “real inner product” ( $\mathbf{x} \cdot \mathbf{y}$ ) does not satisfy the norm property for complex vectors.  
The norm property is that for all vectors,  $|\mathbf{x}|^2 = \langle \mathbf{x}, \mathbf{x} \rangle \geq 0$  and  $|\mathbf{x}| = 0$  only if  $\mathbf{x} = \mathbf{0}$ .  
Hint: What happens for the vector  $\mathbf{x} = (1, i)^T$ ?
- (b) Let  $\mathbf{x} = (a + ib, c + id)^T$ , where  $a, b, c, d$  are real numbers. Show that the “complex inner product” is a norm, with  $|\mathbf{x}|^2 = \langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ .
- (c) How is the value of the complex inner product  $\langle \mathbf{x}, \mathbf{y} \rangle$  related to the value of  $\langle \mathbf{y}, \mathbf{x} \rangle$ ?
- (d) If  $\mathbf{A}$  is a matrix with complex entries, what is the formula for the adjoint  $\mathbf{A}^*$  with respect to the complex inner product? How are the adjoint eigenvalues  $\gamma_k$  related to the  $\lambda_k$  of  $\mathbf{A}$ ?
- (e) Find  $\{\lambda_k, \phi_k, \psi_k\}$  for  $\mathbf{A} = \begin{pmatrix} i & -1 \\ 2 & i+2 \end{pmatrix}$  and show that  $\phi_1 \perp \psi_2$  and  $\phi_2 \perp \psi_1$ .
- (f) Problem 5.5A.6, page 189.

3. (Linear algebra with a different inner product)
- (a) Problem 5.5A.3, page 189.
  - (b) Find the adjoint eigenvectors of  $\mathbf{A}$  with respect to the regular dot product. Verify the orthogonality of  $\phi_i \cdot \psi_j$ .
  - (c) Let  $\mathbf{M} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}$ . Show that the book's " $\mathbf{a} \cdot \mathbf{b}$ " dot product can be written as a "weighted inner product" defined by  $\langle \mathbf{a}, \mathbf{b} \rangle \equiv \mathbf{a} \cdot \mathbf{M}\mathbf{b}$ .
  - (d) Find the positive diagonal matrix  $\mathbf{C}$  such that  $\mathbf{M} = \mathbf{C}^2$ .
  - (e) Multiply the eigenvalue equation  $\mathbf{A}\phi = \lambda\phi$  on the left by  $\mathbf{M}$  to get  $\mathbf{M}\mathbf{A}\phi = \lambda\mathbf{M}\phi$ . Write  $\mathbf{M} = \mathbf{C}^2$  and  $\phi = \mathbf{C}^{-1}\mathbf{y}$  in this equation. Show that the vectors  $\mathbf{y}$  are eigenvectors for  $\mathbf{B}\mathbf{y} = \lambda\mathbf{y}$ , where  $\mathbf{B}$  is a symmetric matrix – Find  $\mathbf{B}$ .
  - (f) Explain why the orthogonality relation for the  $\mathbf{y}$ 's is  $\mathbf{y}_1 \cdot \mathbf{y}_2 = 0$  and use it to justify the orthogonality of the  $\phi$ 's in the weighted inner product from part (c). Hint: use  $\mathbf{y} = \mathbf{C}\phi$  in the dot product.
4. (Solution of initial value problems for matrix-vector ODE systems)  
Problem 5.5A.4, part (a), page 189.
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