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Cohomology of Locally Symmetric Spaces

Leslie Saper, Duke University

http://www.math.duke.edu/faculty/saper/

Theme: Analysis can detect Topology

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However if γ is the counter-clockwise oriented unit circle, one can directly calculate the integral is 2π.

Thus the integral can detect the puncture in $\mathbb{R}^2 \setminus \{(0,0)\}$.

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but not *exact*,

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This closed non-exact form detects the topological fact that $\mathbb{R}^2 \setminus \{(0,0)\}$ has a hole.

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$d_k \colon A^k(X) \to A^{k+1}(X)$ (exterior differentiation),

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$$H^k_{\mathsf{dR}}(X; \mathbb{C}) = \frac{\operatorname{Ker} d_k}{\operatorname{Im} d_{k-1}} \text{ (the } k^{\text{th}} \text{ de Rham cohomology)}.$$

We also have a topological invariant:

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Sketch of Proof: The Poincaré lemma show they agree locally,

$$H^k(U;\mathbb{C}) \cong H^k_{\mathsf{dR}}(U;\mathbb{C}) \cong \begin{cases} \mathbb{C} & k = 0, \\ 0 & k \neq 0. \end{cases}$$

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Both cohomologies are determined by local calculations (Mayer-Vietoris sequence), so the theorem follows. □

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As a first step we need to generalize the Hodge theorem . . .

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Proof:

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Proof:

$$\begin{aligned}\operatorname{Ker} d \,/ \operatorname{Im} d &= \Big(\operatorname{Ker} d \Big/ \overline{\operatorname{Im} d}\Big) \bigoplus \Big(\overline{\operatorname{Im} d}\Big/ \operatorname{Im} d\Big) \\ &= \mathcal{H}_{(2)}(X;\mathbb{C}) \bigoplus \left(\begin{array}{c} 0 \text{ or} \\ \infty\text{-dimensional} \end{array} \right). \qquad \square\end{aligned}$$

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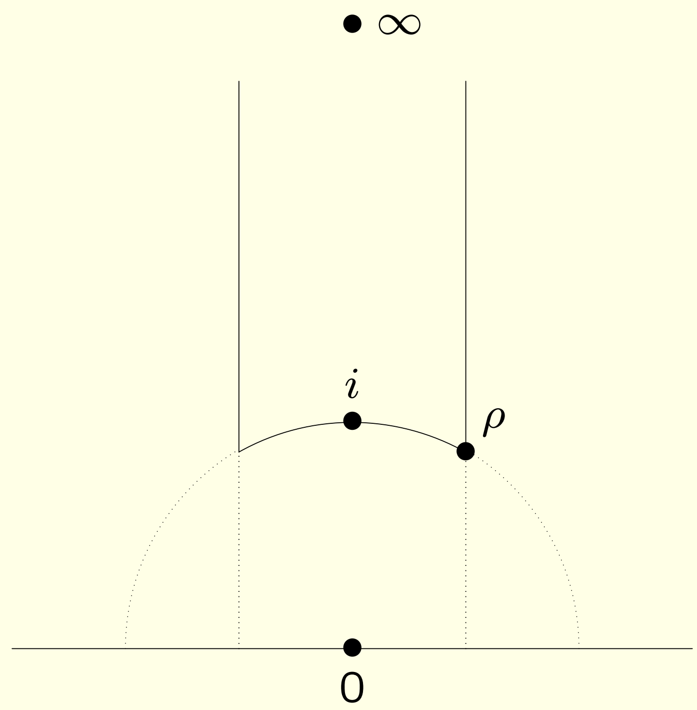

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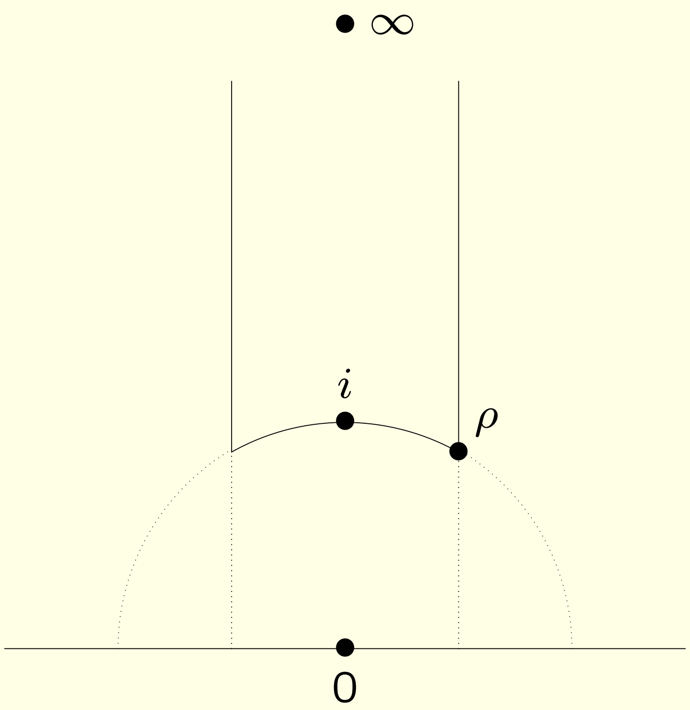

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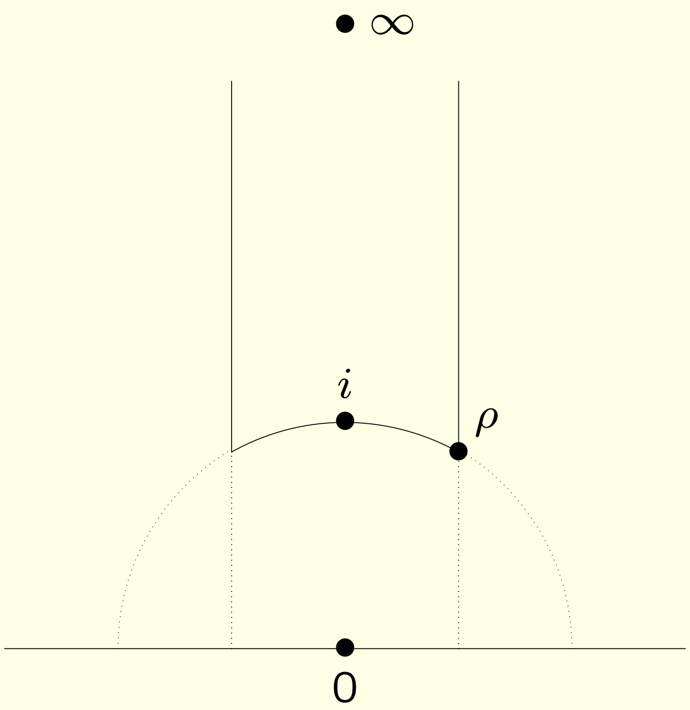

$\mathcal{S}_{k+2}(\Gamma) =$ classical holomorphic modular cusp forms of weight $k+2$, that is, $f\colon H \to \mathbb{C}$ holomorphic,

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{k+2} f(z) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

and f vanishes at all cusps.

In this example the L^2-harmonic 1-forms are known:

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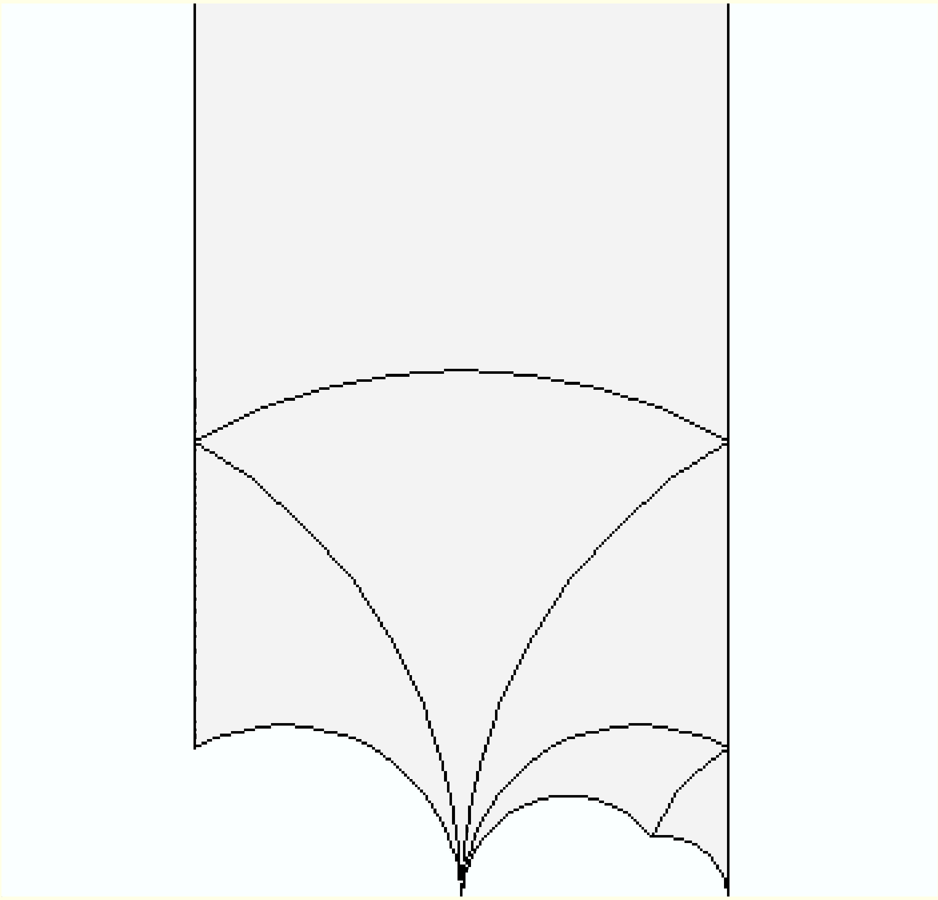
Thus $\theta(\tau)$ is a modular form of weight 2 for $\Gamma_0(4) = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right) \rangle$.

A fundamental domain for $\Gamma_0(4) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \rangle$**:**



(Picture created by *Fundamental Domain drawer*, `http://www.math.lsu.edu/~verrill/fundomain/index2.html`, by Helena Verrill.)

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Jacobi's result follows from the Fourier expansion of $G(\tau)$.

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Rational Structure: $\mathcal{S}_{k+2}(\Gamma)$ is endowed with a rational structure.

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Question: What replaces X^* and H^1_P?

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Baily and Borel show that X^* is a (generally singular) projective algebraic variety.

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Intersection cohomology was introduced by Goresky and MacPherson in order to restore Poincaré duality to the cohomology of singular spaces.

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In view of the fact that X^* is naturally defined over a number field, this result is important for Langlands's program.

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For Y smooth, one obtains the *Hasse-Weil zeta function* of Y, which encodes $\#Y(\mathbb{F}_{p^k})$ for all prime powers p^k.

Thus Langlands's program predicts

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- $Y = X^*$: our case.

Compare L-functions via fixed-point formulas:

Lefschetz fixed point formula for Frobenius on $I_pH(X^*;\mathbb{E})$	$\overset{?}{\longleftrightarrow}$	Arthur-Selberg trace formula for Hecke operators on $H_{(2)}(X;\mathbb{E})$

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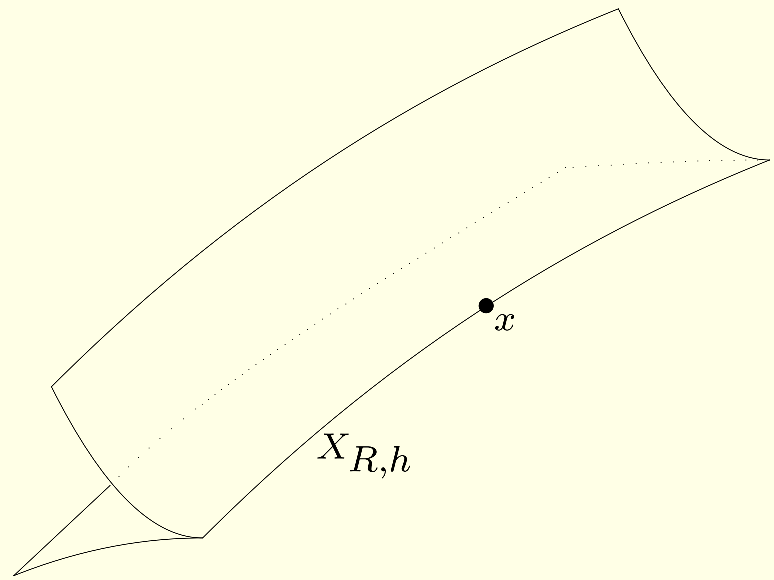
- Many other difficulties.

Structure of X^* near a stratum $X_{R,h}$:



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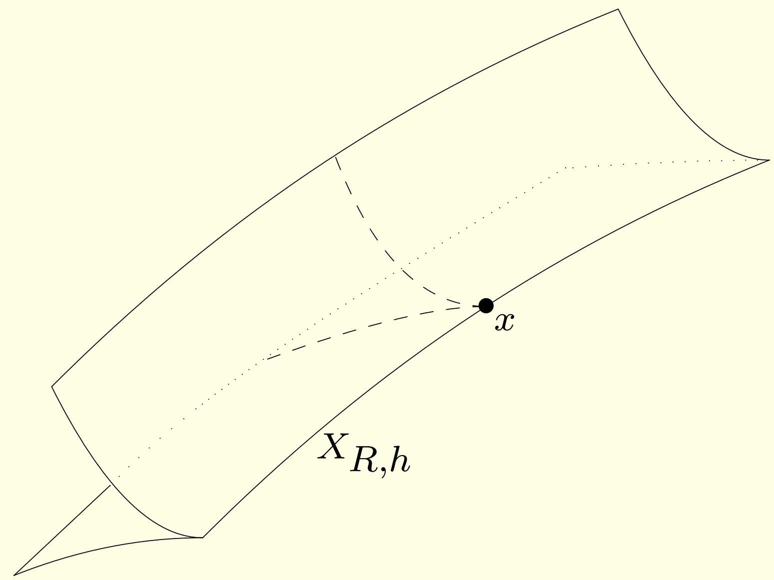

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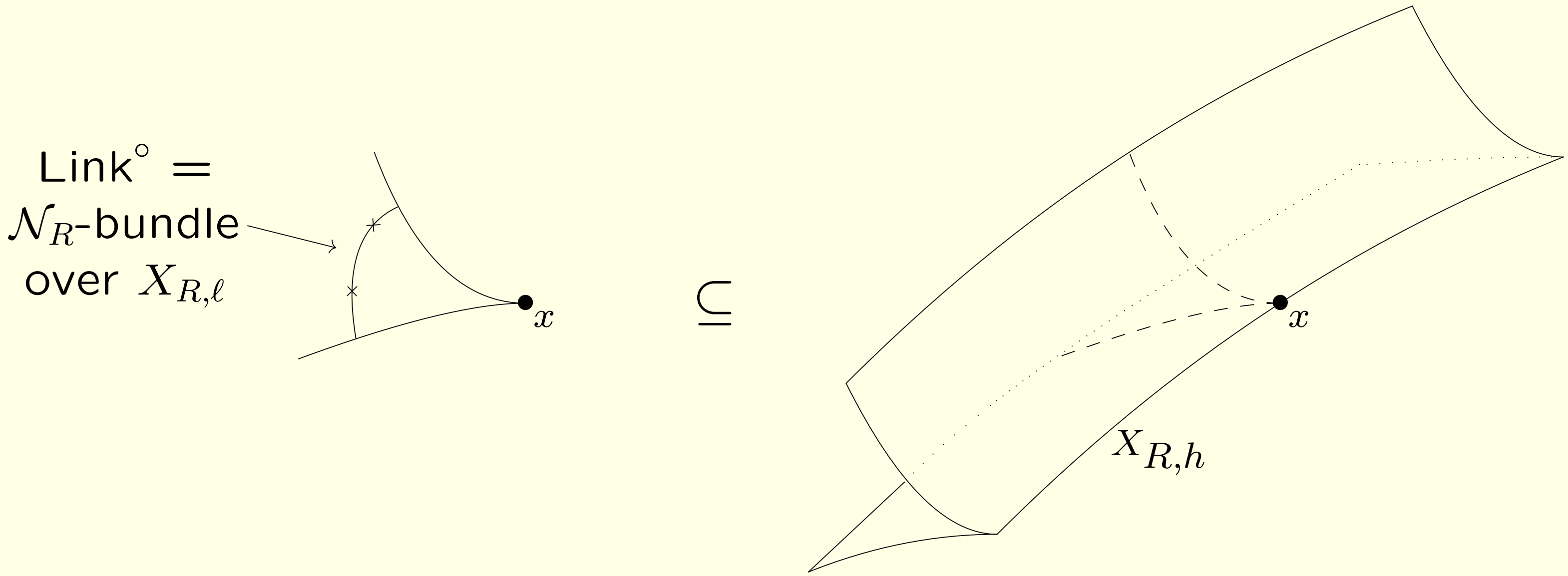

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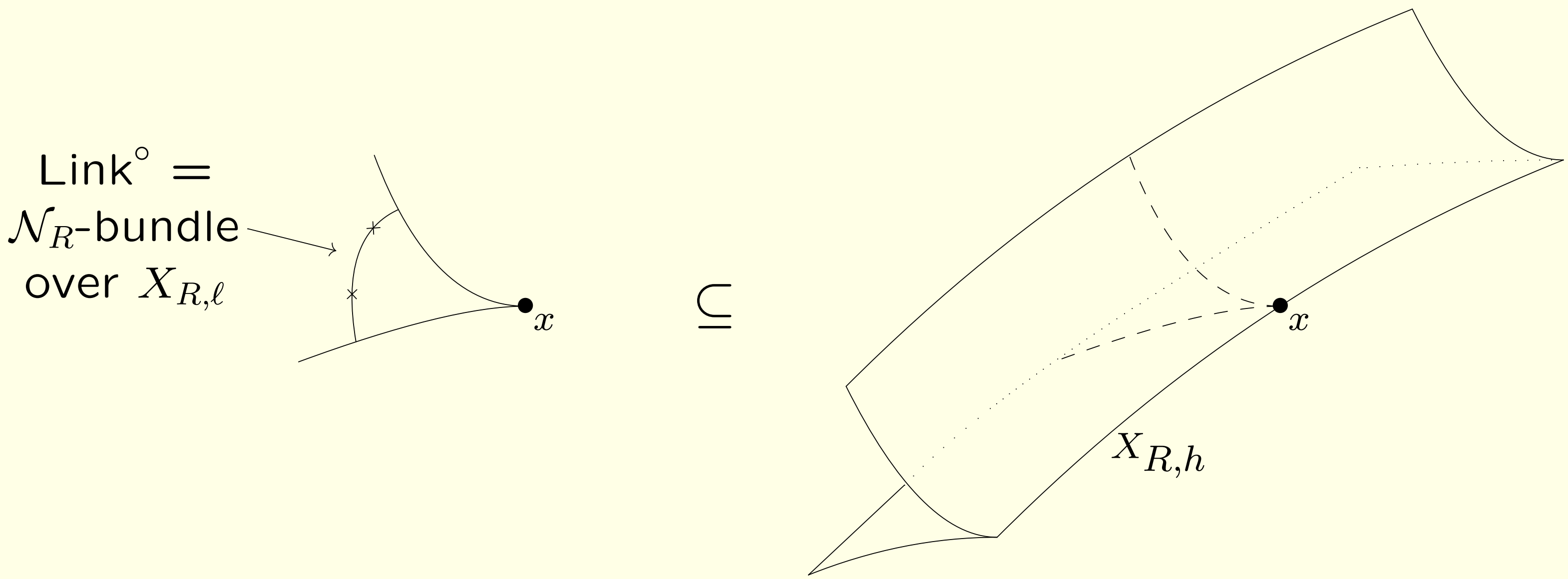
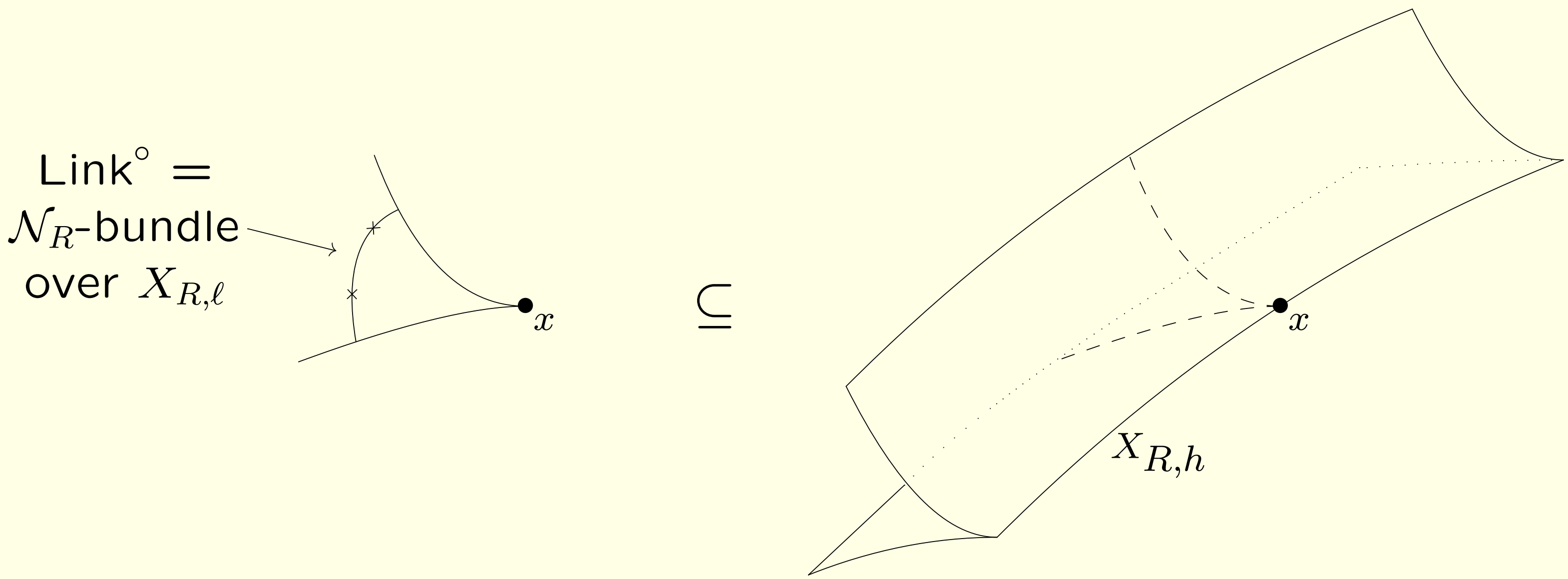


$\mathcal{N}_R = \Gamma \backslash N_R =$ a compact nilmanifold.

$X_{R,\ell} =$ a possibly non-Hermitian locally symmetric space.

Structure of X^* near a stratum $X_{R,h}$:



$\mathcal{N}_R = \Gamma \backslash N_R =$ a compact nilmanifold.
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Since we do not have an effective method in general to compute the cohomology of a locally symmetric space, the local intersection cohomology of X^* is difficult to work with.

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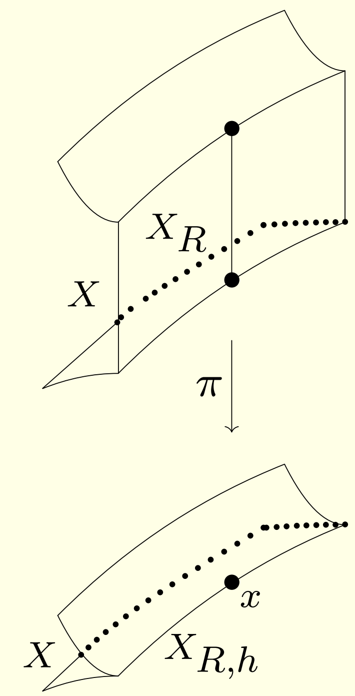
Theorem (S.). *For X Hermitian locally symmetric, $I_pH(X^*;\mathbb{E}) \cong I_pH(\widehat{X};\mathbb{E})$.*

The reductive Borel-Serre compactification $\widehat{X}$

Three constructions:



(i) "Blow up" each stratum of X^* (replace each point with its link) and collapse the nilmanifold fibers

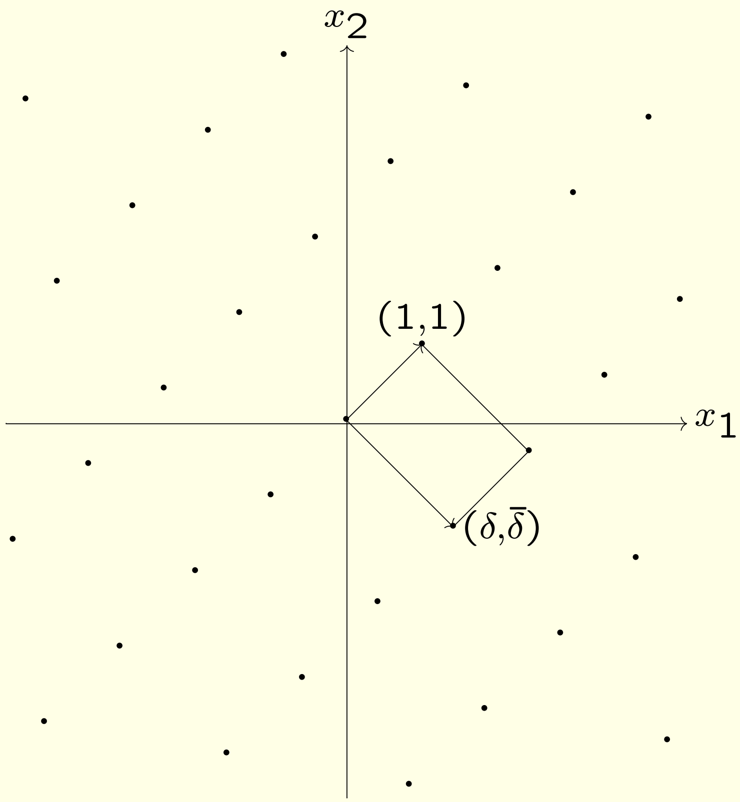
(ii) Remove a neighborhood of each stratum of X^* and collapse the nilmanifold fibers on boundary faces

(iii) Start with the Borel-Serre compactification (1973) $\overline{X}$ (a manifold with corners) and collapse the nilmanifold fibers on the boundary faces (applies to any locally symmetric space X)

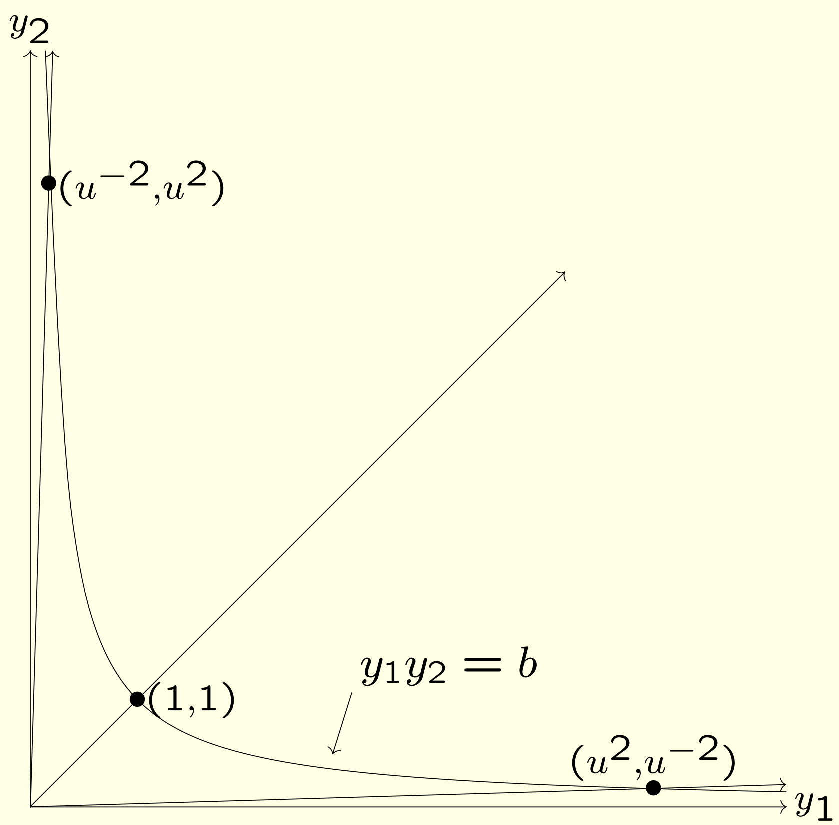
Example: Hilbert Modular Surface $\mathrm{SL}(2,\mathcal{O}_k)\backslash(H\times H)$

Here $k=\mathbb{Q}(\sqrt{d})$, $d>0$. Near "infinity", $\mathrm{SL}(2,\mathcal{O}_k)$ acts via

$$\left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| \, a\in\mathcal{O}_k \right\} \rtimes \left\{ \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \middle| \, u\in\mathcal{O}_k^\times \right\}$$



$\mathcal{O}_k=\mathbb{Z}+\mathbb{Z}\delta$



$\mathcal{O}_k^\times=\{\,u^k \mid k\in\mathbb{Z}\,\}$

Thus

	Boundary stratum	Link
$\overline{X}$	flat T^2-bundle over S^1	point
$\widehat{X}$	S^1	T^2
X^*	point	flat T^2-bundle over S^1

The hyperbola $y_1 y_2 = b$ in the $y_1 y_2$-plane becomes the S^1 above under the action of $\left\{ \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix} \right) \middle| \, u \in \mathcal{O}_k^\times \right\}$. The T^2-fibers correspond to the $x_1 x_2$-plane modulo a lattice.

By the way, the metric is $dr^2 + ds^2_{S^1} + e^{-2r} ds^2_{T^2}$.

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In general, the space S^1 above will be replaced by a locally symmetric space X_P; the fibers T^2 above will be replaced in general by a compact nilmanifold $\mathcal{N}_P$. Here P is a Γ-conjugacy class of parabolic $\mathbb{Q}$-subgroups of G; these index the strata.

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The theory of $\mathcal{L}$-modules and micro-support . . .

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