The Contact Process on Random Graphs and Galton-Watson Trees

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Contact Process on a Graph

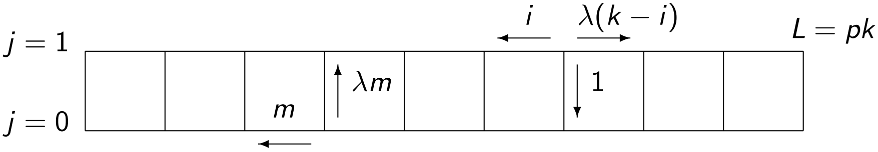
Harris (1974) introduced the contact process on $\mathbb{Z}^d$. Pemantle (1992) was the first to study the contact process on the tree $\mathbb{T}^d$ in which each vertex has degree $d+1$. On a graph it can be defined as follows

The state at time t is $\xi_t \subset G$. The sites in ξ_t are occupied. The others are vacant.

- Occupied sites become vacant at rate 1.
- Vacant sites become occupied at rate λn_1 where n_1 is the number of occupied neighbors.

Liggett's 1999 book *Stochastic Interacting Systems; Contact, Voter, and Exclusion Process* has most of what is known on $\mathbb{Z}^d$ and $\mathbb{T}^d$.

Star Graph

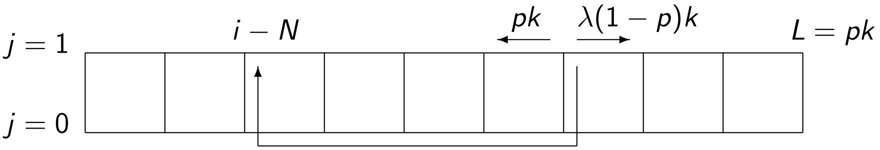


State space is $\{1, \ldots, k\} \times \{0, 1\}$. The first coordinate is the number of occupied leaves, the second is the state of the center.

Let $L = pk$ where $p = \lambda/(1 + 2\lambda)$.

Here N is the number of leaves that die while the center is vacant

$$P(N = j) = \left(\frac{1}{\lambda + 1}\right)^j \cdot \frac{\lambda}{\lambda + 1} \quad \text{for } j \geq 0.$$



We follow Chatterjee and Durrett (2009) and look at the chain only when the center is occupied. The embedded discrete time chain dominates Y_n with transition probability

jump	with prob	note that
$Y_n \to Y_n - 1$	pk/D	$i \le pk$
$Y_n \to \min\{Y_n + 1, pk\}$	$\lambda(1-p)k/D$	$(k-i) \ge k(1-p)$
$Y_n \to Y_n - N$	$1/D$	

N is shifted geometric$(\lambda/(\lambda+1))$

Survival Time on a Star with k leaves

Lower bound. With high probability survives for

$$\geq \exp\left((1-\epsilon)\frac{\lambda^2 k}{2(1+2\lambda)}\right)$$

Upper Bound. Expected survival time

$$\leq (\log k)e^{(1+\epsilon)\lambda^2 k}$$

If $\lambda^2 k \to \infty$ can ignore prefactor.

Intuition. Death is caused by the bad event = "center becomes vacant and does not become occupied before all of the leaves die." The $\log n$ comes from the duration of this event.

Homogeneous Trees $\mathbb{T}_d$, $d > 1$ ($\mathbb{T}_1 = \mathbb{Z}$)

Let 0 be the root of the tree and let P_0 be the probability measure for the process starting from only the root occupied.

$$\lambda_1 = \inf\{\lambda : P_0(\xi_t \neq \emptyset \text{ for all } t) > 0\},$$
$$\lambda_2 = \inf\{\lambda : \liminf_{t\to\infty} P_0(0 \in \xi_t) > 0\}.$$

Pemantle (1992) showed $\lambda_1 < \lambda_2$ if $d \geq 3$ and Liggett (1996) for $d = 2$ by giving bounds on critical values. Stacey (1996) gave a proof that that did not rely on bounds.

DOMath undergrad project, Summer 2018

On the $(1, n)$ tree, in which vertices of degree 1 and n alternate, we have

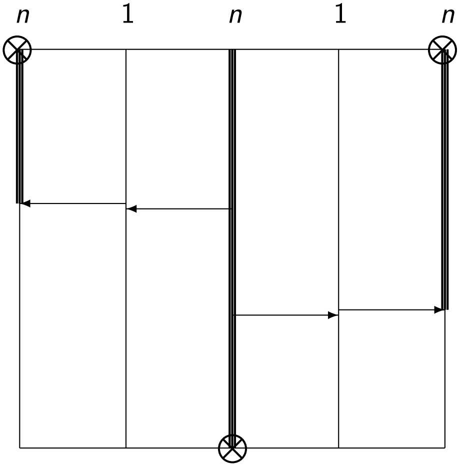
$$\lambda_2 \sim \sqrt{0.5(\log n)/n}.$$

Consider the $(a_1, a_2, \ldots, a_k, n)$ periodic tree with $\max_i a_i \leq n^{1-\epsilon}$ and $a_1 a_2 \cdots a_k = n^b$. As $n \to \infty$ the critical value

$$\lambda_2 \sim \sqrt{\frac{k-b}{2} \cdot \frac{\log n}{n}}$$

Max degree dictates the order; smaller degrees only influence the constant.

Block Construction: Picture



$\geq \epsilon L$ occ. leaves

Block Construction: Details

If block event has probability $\geq 1-\epsilon$, CP survives locally

$\lambda = \sqrt{\theta(\log n)/n}$, $L = n\lambda/(1+2\lambda)$

(i) Have $\geq \epsilon L$ occupied leaves for time

$$\exp(\lambda^2 n) = n^\theta$$

(ii) probability of births $m \to m+1 \to m+2$ in two units of time

$$\geq \delta\lambda^2 \geq C(\log^2 n)/n$$

(iii) whp $m+2$ goes from center occupied to $\geq L(1-\epsilon)$ occupied leaves

Therefore $\theta_c \leq 1+\eta$ for large n

Galton-Watson Trees

Pemantle (1992) *There are constants c_2 and c_3 so that if μ is the mean of the offspring distribution, then for any $k > 1$, if we let $r_k = \max\{2, c_2 \log(1/kp_k)/\mu\}$, μ is mean of offspring dist.*

$$\lambda_2 < c_3 \sqrt{r_k \log r_k \log(k)/k}.$$

If the offspring distribution is a stretched exponential $p_k = c_\gamma \exp(-k^\gamma)$ with $\gamma < 1$ then $\log(1/kp_k) \sim k^\gamma$ and hence $\lambda_2 = 0$.

New Result. $\lambda_2 = 0$ if p_k is subexponential

$$\limsup_{k\to\infty} (1/k) \log p_k = 0.$$

Geometric(p) Offspring Distribution

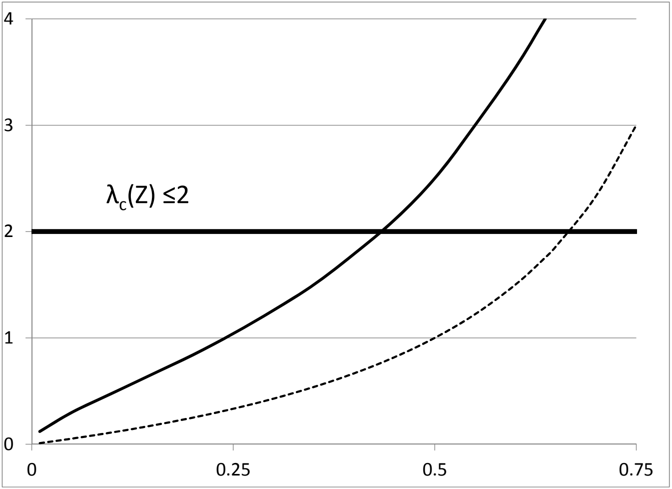


Figure: Upper bounds on λ_1 and λ_2

\$1000 problem

Prove that when the degree distribution is geometric, $\lambda_2 > 0$.

Even better is to prove $\lambda_1 > 0$, but that might be false.

Menard and Singh (2016) have recently proved $\lambda_1 > 0$ for the contact process on random geometric graphs (RGG). A random geometric graph is constructed as follows. Vertices are points of a Poisson process on $\mathbb{R}^d$ with $d \geq 2$. Two vertices $v \neq w$ are connected if $\|v - w\| < R$.

Paul Erdös and ??

Paul Erdös and Terence Tao (age 10)

Finite random graphs

In the early 2000's physicists studied the contact process on random graphs with a power-law degree distribution, i.e., the degree of each vertex is k with probability

$$p_k \sim Ck^{-\alpha} \quad \text{as } k \to \infty.$$

Let β be the critical exponent that controls the rate at which the equilibrium density of occupied sites $\rho(\lambda)$ goes to 0, i.e., $\rho(\lambda) \sim C(\lambda - \lambda_c)^{\beta}$. Mean field calculations gave

- If $\alpha \leq 3$, then $\lambda_c = 0$. If $\alpha < 3$ then $\beta = 1/(3 - \alpha)$.
- If $3 < \alpha \leq 4$, then $\lambda_c > 0$ and $\beta = 1/(\alpha - 3) > 1$.
- If $\alpha > 4$, then $\lambda_c > 0$ and $\beta = 1$.

Chatterjee and Durrett (2009)

Suppose $\alpha > 3$ and $P(d_i \leq 2) = 0$. Condition on the event H that G_n has no self-loops or parallel edges.

Theorem. *Let $\left\{\xi_t^1 : t \geq 0\right\}$ denote the contact process starting from all sites occupied. Then for any $\lambda > 0$, there is a positive constant $p(\lambda)$ so that for any $\delta > 0$*

$$\inf_{t \leq \exp\left(n^{1-\delta}\right)} P\left(\frac{|\xi_t^1|}{n} \geq p(\lambda)\right) \to 1 \quad \text{as } n \to \infty.$$

I.e., $\lambda_c = 0$. They also obtained some bounds on β.

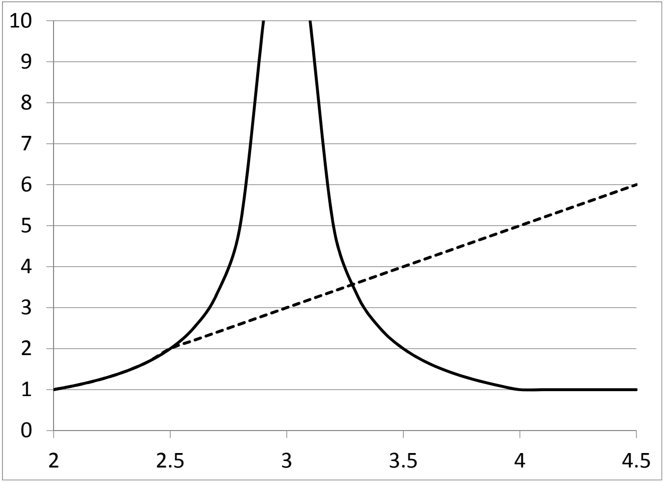
Mountford and friends

In 2013 Mountford, Mourrat, Valesin, and Yao proved upper and lower bounds on $\rho(\lambda)$ with the same dependence on λ, but different constants

$$\begin{cases} \lambda^{1/(3-\alpha)} & 2 < a \leq 5/2 \\ \lambda^{2\alpha-3} \log^{2-\alpha}(1/\lambda) & 5/2 < \alpha \leq 3 \\ \lambda^{2\alpha-3} \log^{4-2\alpha}(1/\lambda) & 3 < \alpha \end{cases}$$

In 2016 Mountford, Valesin, and Yao showed the survival time $\geq e^{cn}$.

Physics versus Rigorous Critical Exponents



Physicists are Never Wrong

In a 2010 paper Castellano and Pastor-Satorras

> *"Already in 2003, Wang et al argued that the SIS epidemic threshold on any graph is set by the largest eigenvalue of the adjacency matrix,* Λ
>
> $$\lambda_c(n) = 1/\Lambda(n).''$$

Two years earlier Pemantle and Stacey proved that $1/\Lambda(n)$ is the **critical value of branching random walk** on the graph. This should not be surprising since the number of paths of length n is $\approx \Lambda^n$.

Maximum eignenvalue

Trivially $\Lambda \geq \sqrt{d_{max}}$

Chung, Lu, and Vu (2003)

$$\Lambda \sim \begin{cases} \langle d^2 \rangle / \langle d \rangle & 2 < \alpha < 5/2 \\ \sqrt{d_{max}} & 5/2 < \alpha \end{cases}$$

where $\langle d^j \rangle$ is the average value of $d(x)^j$ for the graph.

What is Prolonged Persistence?

According to page 942 of the 2015 survey paper in *Reviews of Modern Physics* "Above the epidemic threshold, the activity must be *endemic*, so that the average time to absorption is $O(e^{cn})$."

> *"Chatterjee and Durrett proved that in graphs with power law degree distribution $ET > \exp(O(n^{1-\delta}))$ for any $\delta > 0$. This result pointed to a vanishing threshold, but still left the possibility for nonendemic long-lived metastable states."*

Following the footsteps of Ganesh, Masoulie, and Towsley (2005), we will accept survival for time $\exp(O(n^{\epsilon}))$ for some $\epsilon > 0$ as evidence that $\lambda > \lambda_c$.

Back to the star graph

Wang et al (2003) use the eigenvalue result to conclude that the critical value for the contact process on a star graph with n leaves is $1/\sqrt{n}$.

The results discussed earlier show that the survival time on the star graph increase dramatically when λ changes from $O(1/\sqrt{n})$ to $\gg 1/\sqrt{n}$. However, if $\lambda = \alpha/n$ then for large n the extinction time is

$$\leq Ce^{2\alpha} \log n$$

Since the survival time is $\approx \exp(\lambda^2 n)$ we only have survival for e^{cn} when $\lambda > 0$.

Power law random graphs

Theorem. *Suppose that the degree distribution has*

$$P(d(x) \geq k) = 3^a k^{-a} \quad \textit{for } k \geq 3.$$

We assume $a > 2$ so that $Ed(x)^2 < \infty$. Let $\lambda = n^{-(1-\eta)/2a}$ and $\eta > 0$. If we start from all 1's then there is an $\epsilon > 0$ so that the system survives for time $\exp(O(n^\epsilon))$ with high probability.

$d_{max} = O(n^{-1/a})$ Combining this result with the fact that $1/\Lambda$ gives the critical value for branching random walk we have

$$\lambda_c(n) = n^{-(1+o(1))/2a}.$$

Confirming the eigenvalue formula in this case.

Stretched exponential

$$P(d(x) \geq k) = \exp(-x^{1/b} + 3^{1/b}) \quad \text{for } k \geq 3.$$

where $b > 1$. In this case, $d_{max} \sim \log^b n$, so $1/\Lambda \sim \log^{-b/2} n$

Theorem. Suppose $\lambda_n = \log^{-(1-\eta)(b-1)/2} n$. If we start from all 1's then for any $\epsilon > 0$ the system survives for time $\exp(O(n^{1-\epsilon}))$ with high probability.

We believe our upper bound is accurate. A star has to survive long enough to send offspring to nearby stars. However lower bounds on critical values on graphs with unbounded degree are very difficult.

References

Zoe Huang and Rick Durrett. *The Contact Process on Random Graphs and Galton-Watson Trees.* arXiv 1810.0604

Yufeng Jiang, Remy Kassem, Grayson York, Brandon Zhao, Zoe Huang, Matthew Junge, and Rick Durrett *The contact process on periodic trees.* arXiv 1808.01863

Slides are on my web page. `https://services.math.duke.edu/~rtd/`

Zoe Huang

Figure: Graudation expected in Summer of 2021

Matt Junge

Figure: On the job market now.

Math Research Communities 2019

Stochastic Spatial Models, June 9-15, 2019

Organized by Matt Junge, Shankar Bhamidi, Gerandy Brito, Michael Damron and Rick Durrett. The topics are particle systems on lattices and graphs, first passage percolation, and related problems in random media.

These meetings are not conferences. Most of the time is devoted to working on problems in groups.

Open to ages -2 to 5 (relative to Ph.D.). All 40 accepted participants get their expenses paid (travel, lodging, and meals).

Apply by February 15, 2019 through the AMS web site for the meeting. (google AMS MRC 2019)