Two Particle Systems on Random Graphs

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Joint work with
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Problem 1

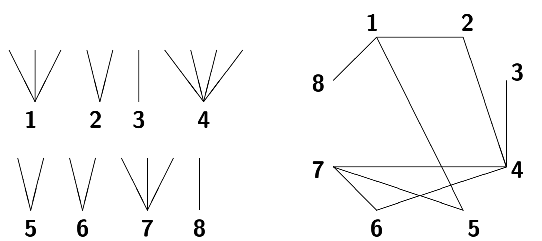
Consider the contact process on a random graph with a power law degree distribution.

Power law random graph. Following Newman, Strogatz, and Watts (2000, 2001) Let $d_1, d_2 \dots$ be i.i.d. with $P(d_i = k) \sim Ck^{-\alpha}$ with $\alpha > 3$ so that $\text{var}\,(d_i) < \infty$.

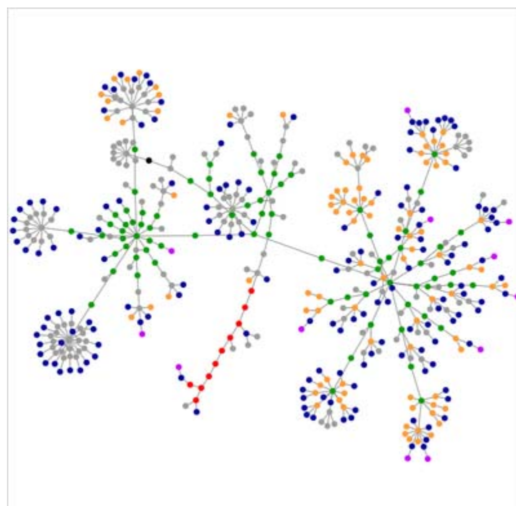
Condition on $\{d_1 + d_2 + \cdots + d_n$ is even$\}$, attach d_i half-edges to vertex i and then pair the half-edges at random. If you want a nice graph then you can condition on the event of positive probability that there is no self-loop and no multiple edges between vertices.

Suppose $P(d_i \le 2) = 0$ so the graph is connected **with high prob**, i.e., with probability $\to 1$ as $n \to \infty$.

Picture of the Construction



Blog network



Contact process

Each site is either healthy 0 or infected 1.

Infected sites become healthy at rate 1.

Healthy sites become infected at rate λ times the number of infected neighbors.

Theorem. Berger, Borgs, Chayes, and Saberi. (2005) *Consider the contact process on the Bárabasi-Albert preferential attachment graph which has* $\alpha = 3$. $\lambda_c = 0$. *With high prob the contact process survives for time* $\ge \exp(Cn^{1/2})$, *and the equilibrium density has*

$$b\lambda^C \le \rho(\lambda) \le B\lambda^c$$

Pastor-Satorras and Vespigiani (2001-2002)

Mean-field theory

Let $\rho_k(t)$ be the fraction of vertices of degree k infected at time t.
$\theta(\lambda) =$ probability a given edge points to an infected site.

$$\frac{d\rho_k(t)}{dt} = -\rho_k(t) + \lambda k(1 - \rho_k(t))\theta$$

so $\rho_k = k\lambda\theta/(1 + k\lambda\theta)$ and

$$\theta = \sum_k q_k \frac{k\lambda\theta}{1 + k\lambda\theta} \quad \text{where} \quad q_k = kp_k/\mu$$

Solve for θ. See Durrett (2007) *Random Graph Dynamics.* 125–128

Mean-field predictions

- If $\alpha \le 3$ then $\lambda_c = 0$.
- If $3 < \alpha < 4$ then $\lambda_c > 0$, $\rho(\lambda) \sim C(\lambda - \lambda_c)^{1/(\alpha-3)}$.
- If $\alpha \ge 4$ then $\beta = 1$.

Generalized to bipartite graphs (think men and women and sexually transmitted diseases) by Gómez-Gardeñes et al. (2008) Proc. Nat. Acad. Sci. 1399-1404

$\lambda_c > 0$ if $\alpha_F, \alpha_M > 3$. (Sweden $\alpha_F = 3.5$, $\alpha_M = 3.3$)

Chatterjee and Durrett, Ann. Prob. submitted

Despite the fact that the graph is locally tree like. Mean-field theory is wrong.

Theorem. *If $\alpha > 3$, $\lambda_c = 0$. With high prob starting from all sites occupied, survives for time $\ge \exp(n^{1-\delta})$ for any $\delta > 0$, and the equilibrium density has*

$$c\lambda^{1+(\alpha-2)(2+\delta)} \le \rho_n(\lambda) \le C\lambda^{1+(\alpha-2)(1-\delta)}$$

When $\alpha > 3$, $1 + (\alpha - 2)(1 - \delta) > 2$, so $\beta > 2$.

Proof also applies to bipartite case.

Keys to the Proof: 1. CP on Stars

Theorem. Let G be a star graph with center 0 and leaves $1, 2, \dots, k$. Let A_t be the set of vertices infected in the contact process at time t. Suppose $\lambda \le 1$ and $\lambda^2 k \ge 50$. Let $L = \lambda k/4$ and let $T = \exp(k\lambda^2/80)/4L$. Let $P_{L,i}$ denote the probability when at time 0 the center is at state i and L leaves are infected. Then

$$P_{L,i}\left(\inf_{t \le T} |A_t| \le 0.4L\right) \le 7e^{-\lambda^2 k/80} \quad \text{for } i = 0, 1.$$

2. Use more than one vertex of high degree

BBCS: In the Bárabasi-Albert model the maximum degree is $O(n^{1/2})$, so the contact process survives for time $O(\exp(n^{1/2}))$.

Look at all of the vertices of degree $\ge n^{\epsilon}$. Infection persists at each for time $\exp(\lambda^2 n^{\epsilon})$

Diameter of graph is $\le C \log n$ so you can push the infection from one star to another in time $\le C \log n$ with probability $\ge n^{-B}$.

Number of infected stars dominates random walk with strong positive drift.

3. Achieving positive density

Use contact process duality with $A = \{x\}$ and $B = G$

$$P(\xi_t^A \cap B \ne \emptyset) = P(\xi_t^B \cap A \ne \emptyset)$$

Survival time for degree k star is $\exp(C\lambda^2 k)$. If we can infect a site with degree $\ge \lambda^{-(2+\delta)}$ it will last a long time.

If we infect a vertex of degree $\ge \lambda^{-(m+\delta)}$ with $m \ge 2$ then the probability we fail to reach one with degree $\ge \lambda^{-(m+1+\delta)}$ is $\le p_m$ where $\sum_m p_m < \infty$.

Bounds on density

Degrees of neighbors have size biased distribution $p_j \sim Cj^{-\alpha+1}$ so

$$\sum_{k > \lambda^{-(2+\delta)}} p_j \sim \lambda^{(\alpha-2)(2+\delta)}$$

extra $+1$ comes from the fact that the site must infect a neighbor before it heals.

In other direction we show that if no vertex of degree $\ge \lambda^{-(1-\delta)}$ nearby dual dies.

Conjecture. *Correct power $1 + 2(\alpha - 2)$ for $\alpha > 3$*

Open problem study $2 < \alpha < 3$. Diameter is $O(\log \log n)$.

Problem 2. Chaos in a spatial epidemic model

Figure: Gypsy moth infestation

Model

Inspiration: In the late 1980s gypsy moths infested the Northeast killing many oak trees. Once the density of gypsy moths got large enough a nuclear polyhedrosis epidemic wiped them out.

G_n is a graph with n vertices. Discrete time = years. Moths lay eggs that hatch the next year. Epidemic spreads quickly

- An occupied site gives rise to a Poisson mean β number of offspring sent to locations chosen at random from the entire graph (local dispersal will be considered later).
- Each site is infected with a small probability α_n. If the site is occupied then the infection spreads and wipes out the connected component of occupied sites containing that vertex.

Deterministic Limit $h(p) = g(f(p))$

Growth. If density of occupied sites is p before growth then density after is

$$f(p) = 1 - e^{-\beta p}$$

Epidemic. Suppose G_n is a random 3-regular graph which looks locally like a tree. Since infection probability is small, only members of the giant component will be killed. If the density before infection is q the density after is

$$g(q) = \begin{cases} q & \text{if } q \in [0, 1/2] \\ (1-q)^3/q^2 & \text{if } q \in [1/2, 1] \end{cases}$$

Theorem. *Suppose $\alpha_n \to 0$ and $\alpha_n \log n \to \infty$. If we start in product measure with density p, densities in process at time $k \geq 0$ on graph converge in probability to $h^k(p)$.*

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Proof. At almost all x, inside the ball of radius $(1/5)\log_2 n$ the random regular graph looks like a tree.

$\alpha_n \log n \to \infty$ guarantees that a cluster escaping from the ball will be hit by infection.

$\beta < 1$ dies out

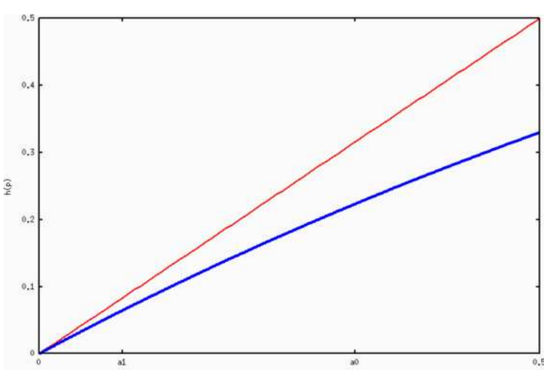


Figure: $\beta = 0.8$

$1 < \beta < 2\log 2$ attracting fixed point

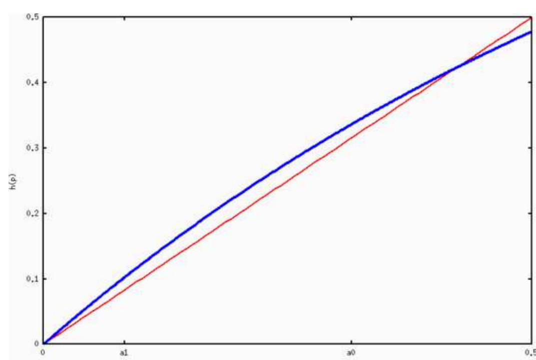


Figure: $\beta = 1.2$

$\beta > 2\log 2$: unstable fixed point

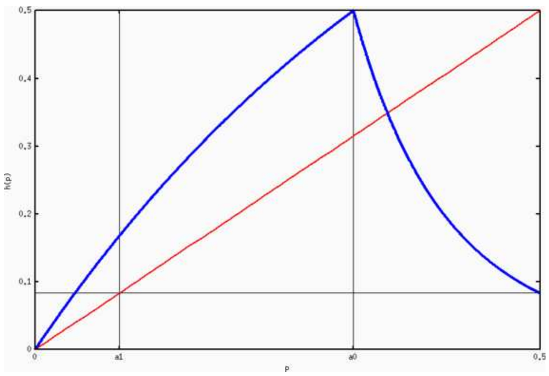
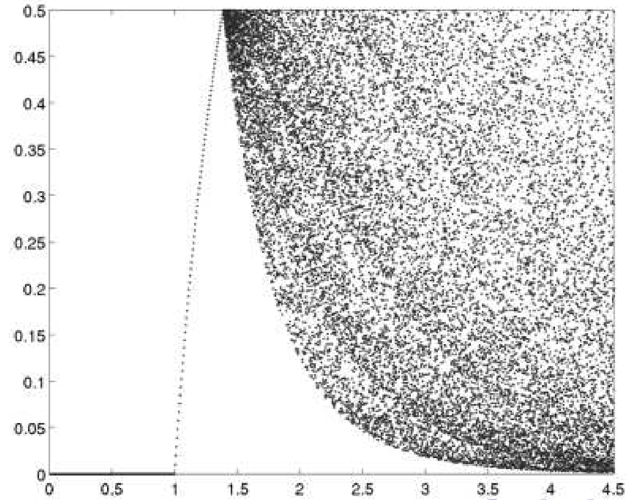


Figure: $\beta = 2\log 3$

Iterates 500 to 550 of h



Period 3 implies chaos

Theorem. Li and Yorke (1975) *If there is a point with* $h^3(c) \le c < h(c) < h^2(c)$ *then*

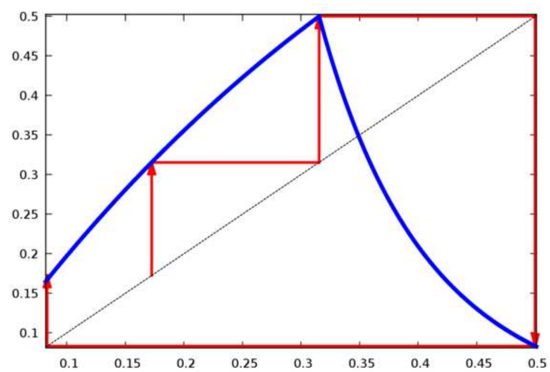
(i) For every k *there is a point with period* k.

(ii) there is an uncountable S *so that if* $p, q \in S$ *and* r *is periodic*

$$\limsup_{N\to\infty} |h^N(p) - h^N(q)| > 0 \qquad \liminf_{N\to\infty} |h^N(p) - h^N(q)| = 0$$

$$\limsup_{N\to\infty} |h^N(p) - h^N(r)| > 0$$

Theorem. *Let* $a_1 = h(1/2)$. *If* $\beta > 2\log 2$ *then the map is chaotic.*



Lasota and Yorke (1973). *There is an absolutely continuous invariant measure if*

$$\inf_{p\in[a_1,1/2]} |(h^n)'(p)| > 1$$

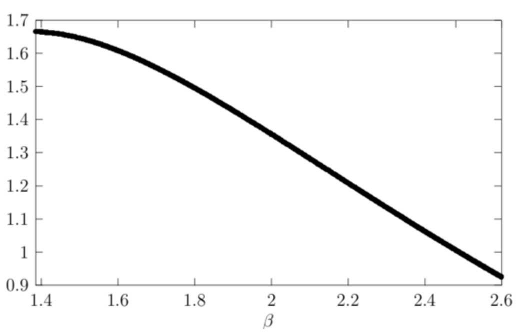
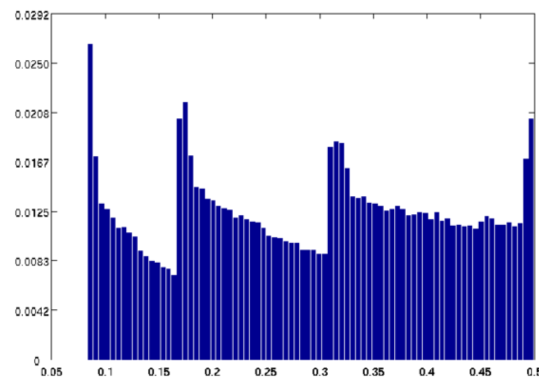


Figure: $n = 3$ condition holds for $\beta \in (2\log 2, 2.48]$

Histogram of orbits for $\beta = 2\log 3$



$G_n = (\mathbb{Z} \bmod L)^2$

Let α_n be epidemic probability and r_n be dispersal range.

Theorem. *If $\alpha_n \to 0$ and $\alpha_n r_n \to \infty$ then densities converge to $h^k(p)$ where $h(p) = g(f(p))$.*

$f(p) = 1 - e^{-\beta p}$

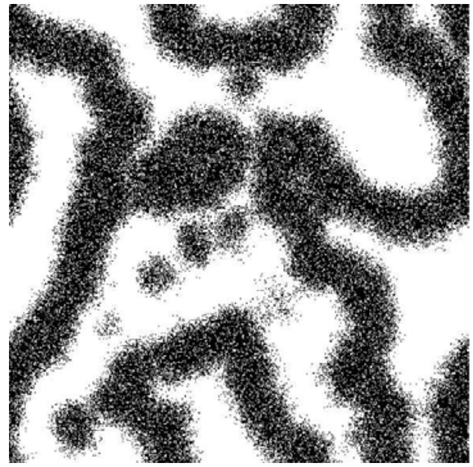
$g(p) = p - P_p(|\mathcal{C}_0| = \infty)$ for site percolation.

Conjecture. *Chaotic for $\beta > \beta_c = \frac{1}{p_c} \log\left(\frac{1}{1-p_c}\right) \approx 1.516$.*

Theorem. *Absolutely continuous invariant measure for $\beta \in (\beta_c, \beta_c + \delta)$.*

Why? $g'(p) \to -\infty$ as $p \downarrow h^{-1}(p_c)$.

Simulation on $\mathbb{Z}^2$ with r_n fixed



Nontrivial Stationary Distribution on $\mathbb{Z}^2$

AE: this is an obvious application of the block construction, so the details should not be added to the paper.

Main idea. After growth the density of the process is $\leq f(1) = 1 - e^{-\beta}$ so even after the most severe epidemic there will be a positive density of sites.

Let $\delta = (1 - e^{-\beta})e^{-4\beta}$. If the neighborhood for growth is $[-L, L]^2$, divide space into square of side $L/2$ and say that the square is occupied if the fraction of occupied sites $> \delta/2$.

Now do a block construction.

Problem 3: Random Boolean Networks

Work in progress with Shirshendu Chatterjee.

Let G_n be a random directed graph in which vertex has in degree $r \geq 3$. (Put r oriented half-edges at each vertex and pair at random).

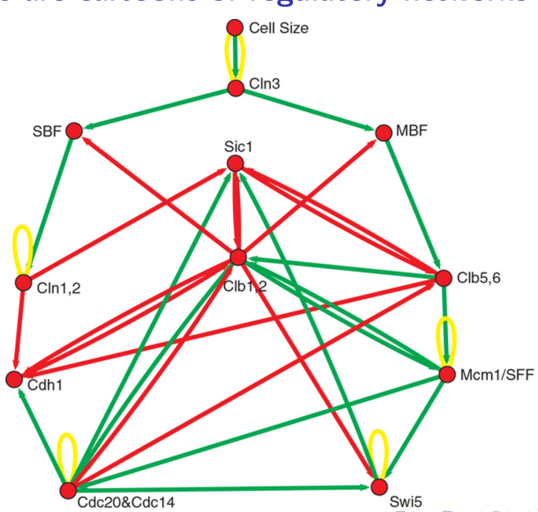
$\eta_n(x) \in \{0, 1\}$ discrete time particle system

Each site x has a random function $\phi_x : \{0,1\}^r \to \{0,1\}$ where the values are i.i.d. and $= 1$ with probability p

$$\eta_{n+1}(x) = \phi_x(\eta_n(y_1(x)), \dots \eta_n(y_r(x)))$$

where the $y_i(x)$ are the vertices with edges pointing to x.

These are cartoons of regulatory networks



Simplified Problem

Think of $\xi_n(x) = 1$ if $\eta_n(x) \neq \eta_{n-1}(x)$

Definition. $\xi_n(x)$ is a threshold voter model in which $\xi_{n+1}(x) = 1$ with probability $q = 2p(1-p)$ if

$$\max_{1 \leq i \leq r} \xi_n(y_i(x)) = 1$$

Why? One of the inputs has changed so the new value will be different from the old with probability $2p(1-p)$

Conjecture. *If $r \geq 3$ prolonged persistence if $qr > 1$.*

Chaos is bad news for a regulatory network. Stuart Kauffman argues that they evolve to the edge of chaos.

Theorem. *If* $q(r-1) > 1$ *then persistence for time* $O(e^{\gamma n})$

Look at the dual which branches from x to all of $\{y_1(x), \dots y_r(x)\}$ with probability q.

Reverse the arrows and let $A^* = \{y : x \to y$ for some $x \in A\}$. This is not the boundary since we may have $y \in A$.

If ϵ is small then with high prob, $|A^*| \geq (r - 1 - \epsilon)|A|$ for all A with $|A| \leq n\epsilon$.

$r - 1$ is sharp. Start with 1 then add a vertex x that points to some y with $1 \to y$, etc.

Theorem. *If* $qr > 1$ *persistence for time* $O(\exp(n^{\beta(q)}))$ *where* $\beta(q) \to 0$ *as* $q \downarrow 1/r$.