ODE and PDE limits for particle systems

Rick Durrett

Evolutionary games with weak selection

The investigation is inspired by two papers. The first is for two strategy games. The second for games with $n \geq 3$ strategies.

Q. When is a strategy favored by selection in a spatial games? I.e., in equilibrium its frequency $> 1/n$.

Tarnita, C.E., Ohtsuki, H., Antal, T., Feng, F., and Nowak, M.A. (2009) Strategy selection in structured populations. *J. Theoretical Biology* 259, 570–581

Tarnita, C.E., Wage, N., and Nowak, M. (2011) Multiple strategies in structured populations. *Proc. Natl. Acad. Sci.* 108, 2334–2337

Evolutionary games: Homogeneously mixing

Given is a game matrix $G_{i,j} \geq 0$ and the frequencies x_j of strategies in the population, $F_i = \sum_j G_{i,j} x_j$ is the fitness of strategy i. Moran model like dynamics: each individual dies at rate 1, and is replaced by an individual chosen at random with probability proportional its fitness. Frequencies of strategies follow the replicator equation

$$\frac{dx_i}{dt} = x_i(F_i - \bar{F})$$

where $\bar{F} = \sum_i x_i F_i$, average fitness

.

Note: If we add a constant to a column of G then $F_i - \bar{F}$ is not changed.

Spatial Model

Suppose space is the d-dimensional integer lattice. **Interaction kernel** $p(x)$ is a probability distribution with $p(x) = p(-x)$, finite range, covariance matrix $\sigma^2 I$. E.g., $p(x) = 1/2d$ for the nearest neighbors $x \pm e_i$, e_i is the ith unit vector.

$\xi(x)$ is strategy used by x. Fitness is $\Phi(x) = \sum_y p(y-x) G(\xi(x), \xi(y))$.

Birth-Death dynamics: Each individual gives birth at rate $\Phi(x)$ and replaces the individual at y with probability $p(y-x)$.

Death-Birth dynamics: Each particle dies at rate 1. Is replaced by a copy of y with probability proportional to $p(y-x)\Phi(y)$. When $p(z) = 1/m$ for a set of neighbors $\mathcal{N}$, we pick with a probaiblity proportional to its fitness.

Small selection

We are going to consider games with $\bar{G}_{i,j} = \mathbf{1} + wG_{i,j}$ where $\mathbf{1}$ is a matrix of all 1's, and w is small. (Selection is small rather than weak since the population size is infinite.)

If the game matrix is $\mathbf{1}$, B-D or D-B dynamics give the voter model. Remove an individual and replace it with a copy of a neighbor chosen at random (according to p). The evolutionary game with small selection is a voter model perturbation in the sense of Cox, Durrett, Perkins (2013) *Astérisque* volume 349, or arXiv:1103.1676

Restrict our attention to $d \geq 3$ so that the voter model has a one parameter family of stationary distributions.

PDE limit

Theorem. Flip rates are those of the voter model $+\epsilon^2 h_{i,j}(0,\xi)$. If we rescale space to $\epsilon\mathbb{Z}^d$ and speed up time by ϵ^{-2} then in $d \geq 3$

$$u_i^\epsilon(t,x) = P(\xi^\epsilon_{t\epsilon^{-2}}(x) = i)$$

converges to the solution of the system of partial differential equations:

$$\frac{\partial u_i}{\partial t} = \frac{\sigma^2}{2}\Delta u_i + \phi_i(u)$$

where the reaction term

$$\phi_i(u) = \sum_{j \neq i} \langle 1_{(\xi(0)=j)} h_{j,i}(0,\xi) - 1_{(\xi(0)=i)} h_{i,j}(0,\xi) \rangle_u$$

and the brackets are expected value with respect to the voter model stationary distribution ν_u in which the densities are given by the vector u.

Key to proof is duality

Voter model is dual to coalescing random walk. $\zeta_s^{x,t}$ is the individual at time $t-s$ who is responsible for the opinion of x at time t. Two lineages that hit coalesce to one.

To handle the perturbation at times of a rate $O(\epsilon^2)$ Poisson process T_n^x, a particle at x branches to include $x+y$ for all y with $p(y)>0$.

The collection of particles $I_s^{x,t}$ is called the influence set. If we know the values in $I_s^{x,t}$ at time $t-s$ then we can compute the value of x at time t.

If we run time at rate ϵ^{-2} the influence set converges to branching Brownian motion. This shows $u(t,x)$ converges. Easy to check it satisfies PDE. See Chapter 2 of CDP.

Birth-Death dynamics

Recall the replicator equation:

$$\frac{du_i}{dt} = \phi_R^i(u) \equiv u_i \left(\sum_k G_{i,k} u_k - \sum_{j,k} u_j G_{j,k} u_k \right).$$

Let v_1, v_2 be independent with distribution p and define random walk coalescence probabilities

$$p_1 = p(0|v_1|v_1 + v_2) \qquad p_2 = p(0|v_1, v_1 + v_2)$$

PDE is $\partial u_i / \partial t = (1/2d)\Delta u + \phi_B^i(u)$ where

$$\phi_B^i(u) = p_1 \phi_R^i(u) + p_2 \sum_{j \neq i} u_i u_j (G_{i,i} - G_{j,i} + G_{i,j} - G_{j,j})$$

Two features of the answer

1. Space enters into the answer through the values of two constants.

$$p_1 = p(0|v_1|v_1 + v_2) \qquad p_2 = p(0|v_1, v_1 + v_2)$$

(Also true for Tarnita's formulas.)

2. ϕ_B is p_1 times the RHS of the replicator equation for the game matrix $G + A$ where

$$A_{i,j} = \frac{p_2}{p_1}(G_{i,i} + G_{i,j} - G_{j,i} - G_{j,j})$$

"The effect of space is equivalent to changing the game matrix." (Ohtsuki and Nowak proved this for the pair approximation.)

Evolutionary games on the torus

$\mathbb{T}_L = (\mathbb{Z} \bmod L)^d$. $N = L^d$. $\bar{G} = \mathbf{1} + wG$. $w = \epsilon^2$

Regime 1. $w \gg N^{-2/d}$.

Run time at rate ϵ^{-2}. Scale space by multiplying by $\epsilon \gg L^{-1}$. Scaled torus converges to R^d and the limit is the PDE we saw on $\mathbb{Z}^d$

Regime 2. $N^{-2/d} \gg w \gg N^{-1}$

In this case the time scale for the perturbation to have an effect, ϵ^{-2} is much larger than the time $O(L^2)$ needed for a random walk to come to equilibrium, but much smaller than the time $O(L^d)$ it takes for two random walks to hit. Because of this, the particles in the dual will (except for times $O(L^2 \log L)$ after the initial time or a branching event) be approximately independent and uniformly distributed across the torus.

Regime 2 limit theorem

$$U_i(t) = \frac{1}{N} \sum_{x \in \mathbb{T}_L} 1\left(\xi^{\epsilon}_{t\epsilon^{-2}}(x) = i\right)$$

Theorem Suppose that $N^{-2/d} \gg w \gg N^{-1}$. If $U_i(0) \to u_i(0)$ then $U_i(t)$ converges uniformly on compact sets to $u_i(t)$, the solution of

$$\frac{du_i}{dt} = \phi_i(u) \qquad u_i(0) = u_i$$

where ϕ_i is the reaction term in the PDE.

Thus in Regime 2, we have "mean-field" behavior, but the reaction function in the ODE is computed using the voter model equilibrium, not the product measure that is typically used in heuristic calculations.

Tarnita's formula

Suppose that in addition to the game dynamics each individual switches to a strategy chosen at random from the n possible strategies at rate μ.

Theorem. Suppose that $N^{-2/d} \gg w \gg N^{-1}$. If $\mu \to 0$ and $\mu/w \to \infty$ slowly enough, then in an n-strategy game strategy k is favored by slection if and only if

$$\phi_k(1/n, \ldots, 1/n) > 0.$$

or $(c_1 G_{k,k} - \bar{G}_{k,*} - \bar{G}_{*,k} - c_1 \bar{G}_{*,*}) + c_2(\bar{G}_{k,*} - \bar{G}) > 0$

Intuitively, in this regime the change from uniformity will be due to lineages that have one branching event. Our result shows that c_1 and c_2 can be expressed in terms of coalescence probabilities.

Configuration model

Let G_n be a graph generated by the **configuration model**. Vertices have degree k with probability p_k. We assign i.i.d. degrees d_i to the vertices and condition the sum $d_1 + \cdots + d_n$ to be even. We attach d_i half-edges to vertex i and then pair the half-edges at random. We will assume that

(A0) the graph G_n has no self-loops or parallel edges.

If $\sum_k k^2 p_k < \infty$ then $P(\text{A0})$ is bounded away from 0 as $n \to \infty$.

(A1) $p_m = 0$ for $m > M$, i.e., the degree distribution is bounded.

(A2) $p_k = 0$ for $k \leq 2$, so random walks have good mixing properties.

Latent Voter Model

The Latent Voter Model introduced by Lambiotte, Saramaki, and Blondel in 2009 models the spread of a technology through a social network. If you have just bought a new iPad and see your neighbors Microsoft Surface tablet then you are unlikely to change. We have states 1, 1^*, 2, and 2^*. The number indicates the technology that the individual owns while * indicates they are in a latent state where they will not change their opinion. Our process takes place on a graph generated by the configuration model. Letting f_i be the fraction of neighbors in state I, the transition rates are as follows

$$\begin{array}{ll} 1 \to 2^* \text{ at rate } f_2 & 2^* \to 2 \text{ at rate } \lambda \\ 2 \to 1^* \text{ at rate } f_1 & 1^* \to 1 \text{ at rate } \lambda \end{array}$$

Construction

Each site x has a Poisson process with rate 1. For each arrival we have a random choice of neighbor Y_n^x, $n \geq 1$. At time T_n^x, we draw an arrow from Y_n^x to x to indicate that to indicate that if the individual at x is active (not in state 1^* or 2^*) at time t then they will imitate the opinion at Y_n^x.

We introduce for each site x, a rate λ Poisson process W_n^x, $n \geq 1$ of "wake-up dots" that return the voter to the active state.

- If there is only one voter arrow between two wake up dots, the result is an ordinary voter event.
- If between two wake up dots there are voter arrows to x from two different neighbors, an event of probability $O(\lambda^{-2})$, then x will change its opinion if and only at least one of the two neighbors has a different opinion.

LV as a voter model perturbation

There are $O(\lambda)$ wake up dots at a site in time t. We run time at rate λ so we have some events with two arrows between successive wake-up dots. The probability of three or more arrows between two wake up dots $\to 0$.

If we let $y_1, \dots y_{d(x)}$ be an enumeration of the nearest neighbors of x, the perturbation is

$$h_{1,2}(x,\xi) = 1_{\{\xi_t(x)=1\}} \frac{2}{d(x)^2} \sum_{1 \le k < \ell \le d(x)} 1_{\{\xi(y_k) \text{ or } \xi(y_\ell) \in \{2,2^*\}\}}$$

Similar formulas hold when the roles of 1 and 2 are interchanged.
$h_{1^*,j} = h_{2^*,j} \equiv 0$.

The reaction term is

$$\phi(u) = \langle h_{2,1}(0,\xi) - h_{1,2}(0,\xi) \rangle_u = c_G u(1-u)(1-2u)$$

ODE limit on random graph

Let $\pi(x) = d(x)/D$ be the stationary distribution for the reandom walk.

$$U^n(t) = \sum_x \pi(x) 1_{\{\xi_{\lambda t}(x)=1\}}$$

Theorem. *Suppose that* $\log n \ll \lambda_n \ll n$. *If* $U^n(0) \to u_0$ *then* $U^n(t)$ *converges in probability and uniformly on compact sets to* $u(t)$*, the solution of*

$$\frac{du}{dt} = c_G u(1-u)(1-2u) \qquad u(0) = u_0.$$

$\log n \ll \lambda_n$ implies that random walks will randomize their positions between non-voter events. $\lambda_n \ll n$ since two random walks take time $O(n)$ to hit.

Long time survival on random graph

The latent voter model has two absorbing states $\equiv 1$ and $\equiv 2$ so on a finite graph it will eventually reach one of them. However, by analogy with the contact process on the torus and or on power-law random graphs, we expect survival for time $\exp(\gamma n)$ for some $\gamma > 0$.

Theorem. *Suppose that* $\log n \ll \lambda_n \ll n$. *Let* $\epsilon > 0$ *and* $m < \infty$. *If* $U^n(0) \to u_0 \in (0,1)$ *there is a* $T_0(\epsilon)$ *that depends on the initial density so that for any* $m < \infty$ *if* n *is large then with high probability*

$$|U^n(t) - 1/2| \le \epsilon \qquad \text{for all } t \in [T_0(\epsilon), n^m].$$

The result is proved using ideas from Darling, R.W.R., and Norris, J.R. (2008) Differential equation approximation for Markov chains. *Probability Surveys.* 5, 37–79

References

Ted Cox, and Rick Durrett (2016) Evolutionary games on the torus with weak selection. *Stoch. Proc. Appl.* 126, 2388-2409

Ran Huo and Rick Durrett (2018) Latent Voter Model on Locally Tree-Like Random Graphs. *Stoch. Proc. Appl.* 128, 1590-1614

Both papers and the slides for this talk are on my web page.