Evolving voter model

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A result of the 2010-2011 SAMSI program on complex networks

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Holme and Newman (2006)

They begin with a network of N nodes and M edges, where each node x has an opinion $\xi(x)$ from a set of G possible opinions and the number of people per opinion $\gamma_N = N/G$ stays bounded as N gets large.

On each step a vertex x is picked at random. If its degree $d(x) = 0$, nothing happens. If $d(x) > 0$,

(i) with probability α an edge attached to vertex x is selected and the other end of that edge is moved to a vertex chosen at random from those with opinion $\xi(x)$.

(ii) otherwise (i.e., with probability $1 - \alpha$) a random neighbor y of x is selected and the opinion of x is set to $\xi(y)$.

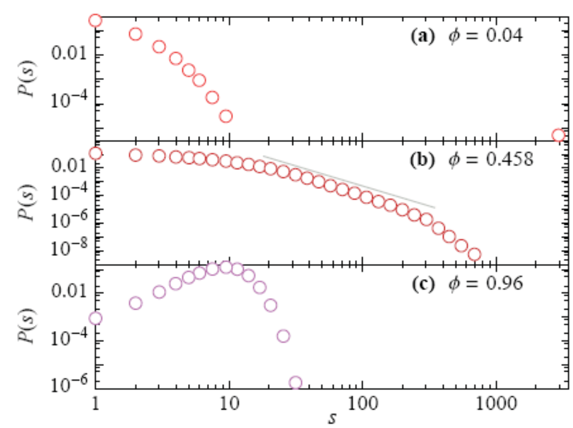
Eventually there are no edges that connect different opinions and the system freezes at time τ_N.

Extreme Cases

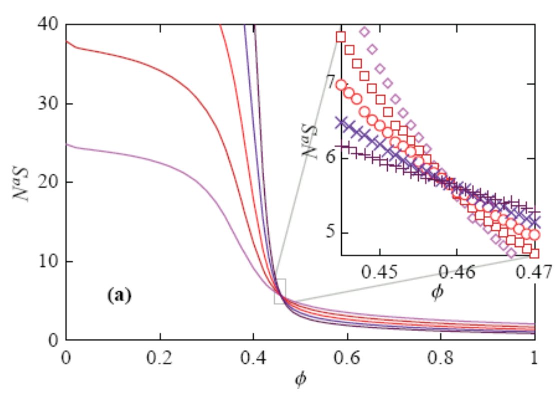
When $\alpha = 1$ **only rewiring steps occur**, so once all of the M edges have been touched the graph has been disconnected into G components, each of which is small. By results for the coupon collector's problem, $\tau_N \sim M \log M$ updates.

When $\phi = 0$ **this is a voter model on a static graph**. If we use an Erdös-Renyi random graph in which each vertex has average degree $\lambda > 1$ then there is a giant component with a positive fraction of the vertices and a large number of small components with size $O(\log N)$. The giant component will reach consensus after $\tau_N = O(N^2)$ updates, so the end result is one opinion with a large number of followers while all of the other populations are small.

Community sizes $N = 3200$, $M = 6400$, $\gamma = 10$.



Finite size scaling



Our model: continuous time

Events happen on each oriented edge (x, y) at times of a rate one Poisson process. **(Isothermal voter model.)** N^2 updates $\to$ time N.

If the voters at the two ends of the edge agree then we do nothing.

If they disagree, then with probability $1 - \alpha$ the voter at x adopts the opinion of the voter at y.

With probability α, x breaks its connection to y and makes a new connection to a voter chosen at random:

(i) from all of the vertices in the graph **"rewire to random"**,

(ii) from those that share its opinion **"rewire to same."**

Opinions $\{0, 1\}$. Initial state product measure with density u.

Rewire to random with $u = 1/2$

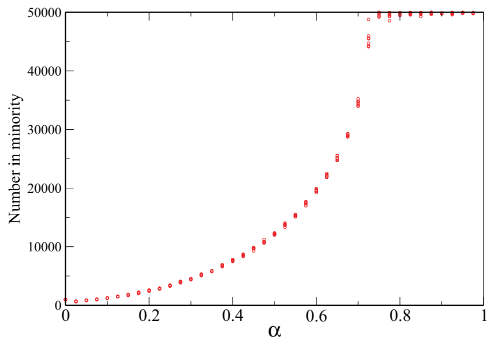


Figure: Erdos-Renyi, $\lambda = 4$, $N = 100,000$

Rewire to random: A universal curve?

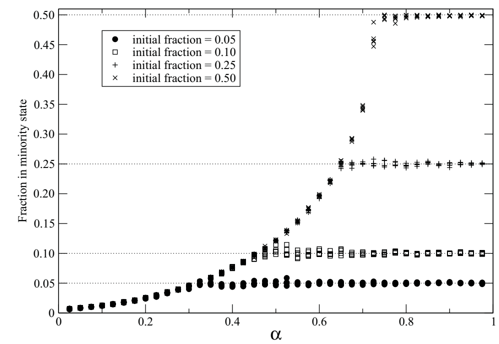
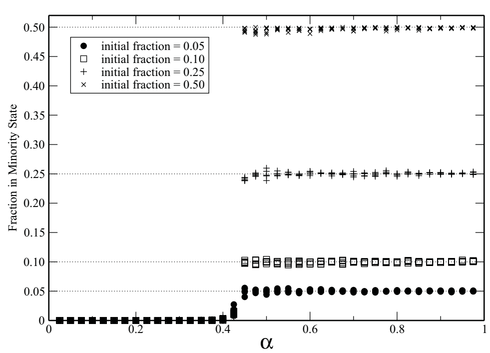


Figure: Critical value $\alpha_c(u)$ depends on u, but when $\alpha < \alpha_c(u)$ minority fraction agrees with curve for $u = 1/2$.

Rewire to same: discontinuous transition



The simulation that showed us the answer

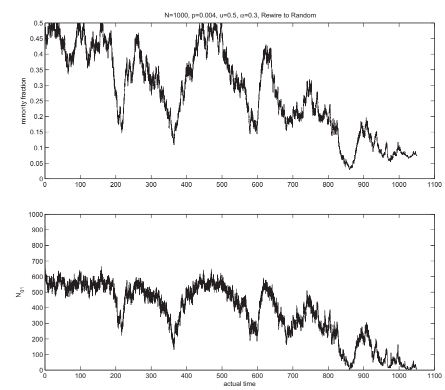


Figure: $N = 1000$, $\lambda = 4$, $u = 1/2$, Initial $N_{10} = 1000$.

Holley and Liggett (1975)

Consider the voter model on the d-dimensional integer lattice $\mathbb{Z}^d$ in which each vertex decides to change its opinion at rate 1, and when it does, it adopts the opinion of one of its $2d$ nearest neighbors chosen at random.

In $d \leq 2$, the system approaches complete consensus. That is if $x \neq y$ then $P(\xi_t(x) \neq \xi_t(y)) \to 0$.

In $d \geq 3$ if we start from ξ_0^p product measure with density p, i.e., $\xi_0^p(x)$ are independent and equal to 1 with probability then ξ_t^p converges in distribution to a limit ν_p, which is a stationary distribution for the voter model.

Cox and Greven (1990)

The voter model on the torus in $d \geq 3$ at time Nt then it locally looks like $\nu_{\theta(t)}$ where the density changes according to the Wright-Fisher diffusion:

$$d\theta_t = \sqrt{\beta_d \cdot 2\theta_t(1-\theta_t)}dB_t$$

The quantity under the square root is the fraction of discordant edges under $\nu_{\theta(t)}$.

There is a one parameter family of quasi-stationary distributions, and the parameter changes according to a diffusion.

quasi-stationary since in the finite voter model all 0's and all 1's are absorbing.

N_{01} versus N_1, $\alpha = 0.5$

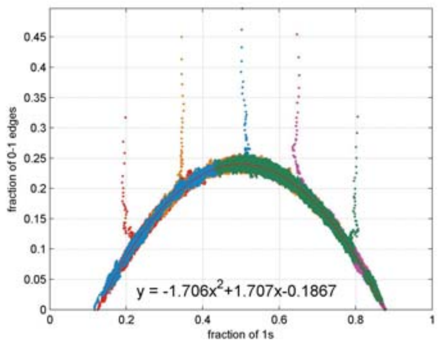
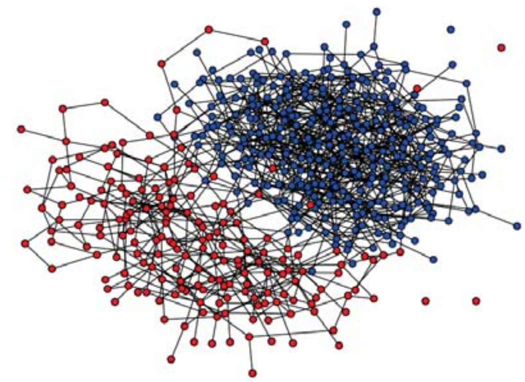


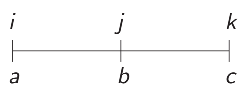
Figure: Process comes quickly to the arch then diffuses along it, splitting into two when it reaches the end.

Graph fission for $\alpha = 0.65$



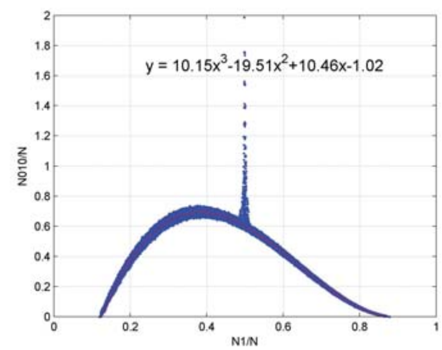
Finite dim. distr. on a random graph

A definition from the theory of graph limits of Lovasz et al. N_{ijk} is the number of homomorphisms from the labeled graph



into our labeled graph (G, ξ). When $i = 0$, $j = 1$, $k = 0$ every triple is counted twice but this seems like the natural definition.

N_{010} versus N_1, $\alpha = 0.5$



N_{ijk} are polynomials?

Bill Shi's simulations for $\lambda = 4$, $\alpha = 0.5$

$$\begin{aligned}
N_{01} &= -3.42x^2 + 3.42x - 0.38 \\
N_{110} &= -13.53x^3 + 10.87x^2 + 1.19x - 0.30 \\
N_{001} &= 13.54x^3 - 29.74x^2 + 17.67x - 1.77 \\
N_{101} &= -10.14x^3 + 10.93x^2 - 1.89x + 0.08 \\
N_{010} &= 10.15x^3 - 19.51x^2 + 10.46x - 1.02
\end{aligned}$$

$N_{110}(x) = N_{001}(1-x)$, $N_{101}(x) = N_{010}(1-x)$

Evolution Equations

$$\begin{aligned}
\frac{dN_{10}}{dt} &= -(2-\alpha)N_{10} + (1-\alpha)[N_{100} - N_{010} + N_{110} - N_{101}] \\
\frac{1}{2}\frac{dN_{11}}{dt} &= (1-\alpha(1-u))N_{10} + (1-\alpha)[N_{101} - N_{011}] \\
\frac{1}{2}\frac{dN_{00}}{dt} &= (1-\alpha u)N_{10} + (1-\alpha)[N_{010} - N_{100}]
\end{aligned}$$

Of course $N_{11} + 2N_{10} + N_{00} = M$, the number of edges.

$$\sum_{ijk} N_{ijk} = \sum_y d(y)(d(y)-1)$$

$$\frac{d}{dt}\sum_{ijk} N_{ijk} = -2\alpha[N_{101} + N_{010} + N_{100} + N_{110}] + 4\alpha N_{10} \cdot \frac{M}{N}$$

One equation short

When $\lambda = 4$, $\alpha = 0.5$

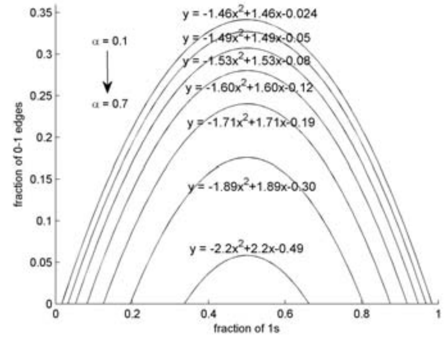
	from equations	from simulation
N_{101}/N_{01}	$(2a_3 + 2b_3) - 2b_3 u$	$-0.23 + 2.96u$
N_{010}/N_{01}	$2a_3 + 2b_3 u$	$2.73 - 2.96u$
N_{110}/N_{01}	$(2a_3 + 2b_3 + 1) - (2b_3 - 1)u$	$0.77 + 3.96u$
N_{100}/N_{01}	$(2a_3 + 2) + (2b_3 - 1)u$	$4.73 - 3.96u$

From $(d/dt) \sum_{ijk} N_{ijk} = 0$ we get

$$2\lambda(1 - \alpha) = 4a_3 + 2 + 2b_3 - \alpha$$

Equations and simulation agree if $2a_3 = 2.73$ and $2b_3 = -2.96$.

Arches for rewire to random



Arches for rewire to same

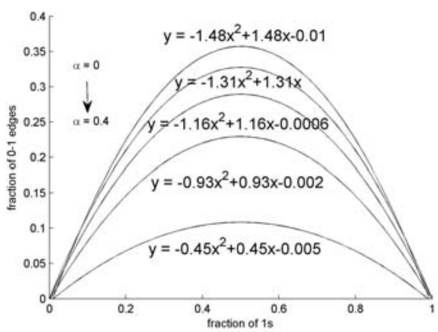


Figure: Note that constant term ≈ 0, explaining discontinuous distribution.

Why do the arches behave differently in the two versions of the model?

Extensions

We get the same result if we start with a random 4-regular graph

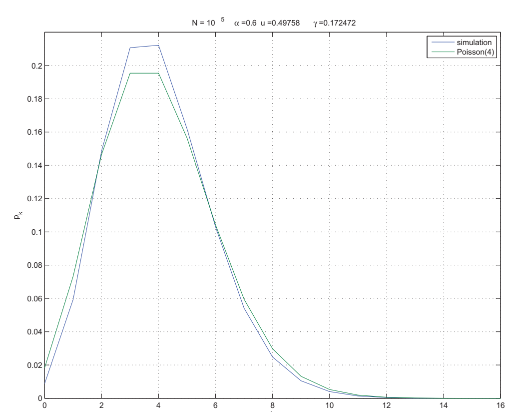
OR

if we designate uN vertices as 1 and $(1-u)N$ as 0 and connect an i node to a j node with probability p_{ij}/N.

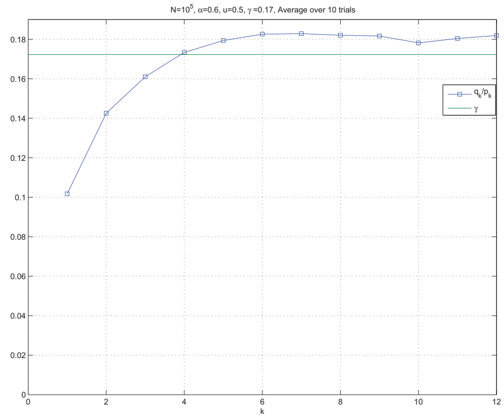
In the second case by choosing the p_{ij} correctly we can achieve any possible value of N_1/N and N_{10}/M where $M = \lambda N/2$.

For these initial conditions we quickly move to the arch of quasistationary distributions.

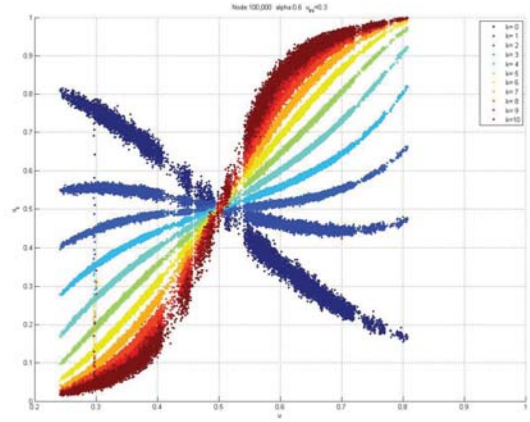
Degree distribution Poisson?



Poisson iff fraction of $\neq$ neighbors is constant



Fraction of degree k nodes $= 1$



Open Problems

Prove quasi-stationary distributions exist when α is small.

When $\alpha = 0$ we know the quasi-stationary distributions for the voter model, so it is natural to try a perturbation argument. However when we consider (G, ξ) for the voter model the G does not change.

Singular perturbation problem.
Work with Jonathan Mattingly and David Sivakoff.

First step: understand $\lim_{\alpha \to 0}$ which is $\neq$ system with $\alpha = 0$.
Answer: rewire, voter model equilibrates, rewire again.

Harder second step: There is a unique stationary distribution on the space of graphs with N vertices and M edges.

Conjecture. *In the rewire to random model if $\alpha < \alpha_c(1/2)$ and $v(\alpha) < u \le 1/2$ then starting from product measure with a density u of 1's, the evolving voter model converges rapidly to a quasi-stationary distribution $\nu_{\alpha,u}$. At time tN the evolving voter model looks locally like $\nu_{\alpha,\theta(t)}$ where the density changes according to a generalized Wright-Fisher diffusion process*

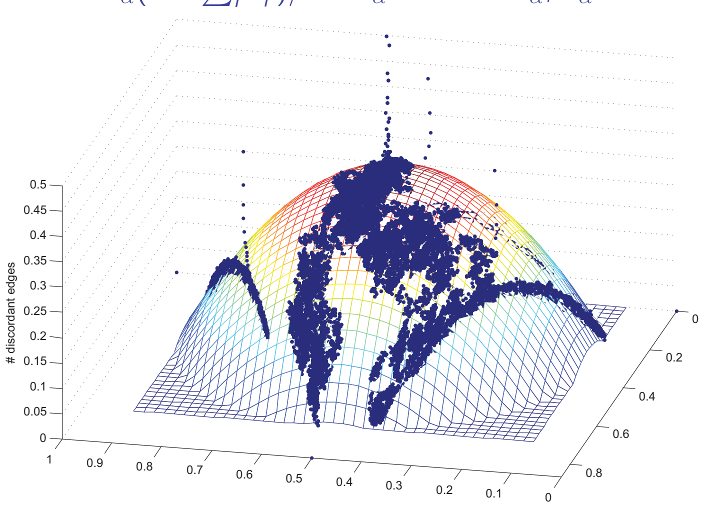
$$d\theta_t = \sqrt{(1-\alpha)[c_\alpha \cdot \theta_t(1-\theta_t) - b_\alpha]}dB_t$$

until θ_t reaches $v(\alpha)$ or $1 - v(\alpha)$.

Rewire to same is similar but $b_\alpha = 0$.

What happens with more than two initial types?

Arch is $c_\alpha(1 - \sum_i u_i^2)/2 - b_\alpha$ for same c_α, b_α?



c_α and b_α as a function of $\beta = \alpha/(1-\alpha)$

Generator $L = V + \beta R$, **V**oter, **R**ewire

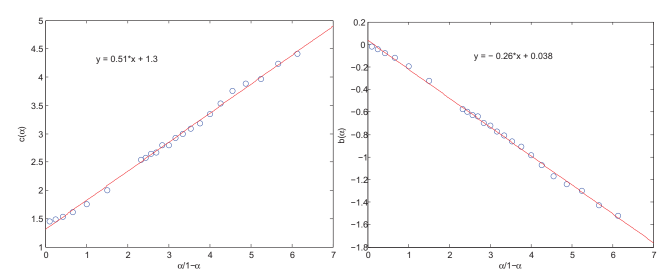


Figure: Conjecture. $c_\alpha = a + 2b\beta$, $b_\alpha = -b\beta$

Linear in β, not a power series.

Infinitely many phase transitions

Suppose $c_\alpha = a + 2b\beta$, $b_\alpha = -b\beta$, $\beta = \frac{\alpha}{1-\alpha}$

If there are k opinions the smallest value of $1 - \sum_i u_i^2$ is $1 - 1/k$

$N_{\neq} = \frac{c_\alpha}{2}(1 - 1/k) - b_\alpha$

This is 0 when $\beta_k = a(k-1)/2b$. (needs $2b$ and $-b$).

$a = 1.3$, $b = 0.25$
$k = 2$, $\beta_2 = 2.6$, $\alpha_2 = \beta_2/(1+\beta_2) = 0.72$
$k = 3$, $\beta_3 = 5.2$, $\alpha_3 = 0.84$

α_k is the critical value for starting with k types.

If we start with a large number types then for $\alpha < \alpha_2$ we may end up with two or more types at the end, but if $\alpha_2 < \alpha < \alpha_3$ we will always end up with three or more, etc.

Higher order statistics

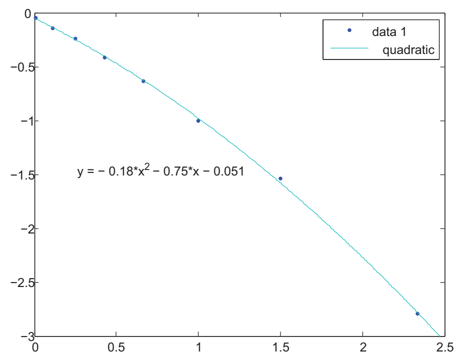
α	$y = N_{010}/N$ versus $x = N_1/N$
0.1	$y = 8.3256x^3 - 16.8145x^2 + 8.6220x - 0.14048$
0.2	$y = 8.3574x^3 - 16.7826x^2 + 8.6622x - 0.24319$
0.3	$y = 8.6960x^3 - 17.2870x^2 + 9.0065x - 0.41525$
0.4	$y = 8.9222x^3 - 17.6819x^2 + 9.3873x - 0.63602$
0.5	$y = 9.9584x^3 - 19.2445x^2 + 10.3545x - 1.0078$
0.6	$y = 11.7247x^3 - 21.7348x^2 + 11.8134x - 1.5414$
0.7	$y = 16.9464x^3 - 29.4114x^2 + 16.2660x - 2.7904$

$\alpha = 0 \quad ax(1-x)^2 + bx(1-x) = ax^3 - (2a+b)x^2 + (a+b)x$

$$a = c_\lambda \bar{p}(x|y|z) \qquad b = c_\lambda \bar{p}(xz|y)$$

$c_\lambda = \sum_x d_x(d_x - 1)/N$, $\bar{p}$ coalesce probs averaged over triples

N_{010} Coefficients Quadratic: Constant Term



$N_{010} - N_{100} = (1 + \beta u)N_{01}$

Exactly solvable model?

Graph statistics N_{01}, N_{010}, N_{100}, etc. are polynomials in u and β

Unfortunately there does not seem to be a dual for the evolving voter model.