# Logarithmic Fluctuations From Circularity 

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## Talk Outline

- Part 1: Logarithmic fluctuations
- Part 2: Limiting shapes
- Part 3: Integrality wreaks havoc
- Part 1: Joint work with David Jerison and Scott Sheffield.
- Parts 2 \& 3: Joint work with Anne Fey and Yuval Peres.


# Part 1: Logarithmic Fluctuations 

## From random walk to growth model

## Internal DLA

- Start with $n$ particles at the origin in the square grid $\mathbb{Z}^{2}$.
- Each particle in turn performs a simple random walk until it finds an unoccupied site, stays there.
- $A(n)$ : the resulting random set of $n$ sites in $\mathbb{Z}^{2}$.

Growth rule:

- Let $A(1)=\{o\}$, and

$$
A(n+1)=A(n) \cup\left\{X^{n}\left(\tau^{n}\right)\right\}
$$

where $X^{1}, X^{2}, \ldots$ are independent random walks, and

$$
\tau^{n}=\min \left\{t \mid X^{n}(t) \notin A(n)\right\} .
$$



# Questions 

- Limiting shape
- Fluctuations


## Meakin \& Deutch, J. Chem. Phys. 1986

- "It is also of some fundamental significance to know just how smooth a surface formed by diffusion limited processes may be."


FIG. 2. Dependence of the variance of the surface height $(\xi)$ on the strip width $l$ for two-dimensional (square lattice) diffusion limited annihilation in the long time ( $\bar{h}>l$ ) limit.

- "Initially, we plotted $\ln (\xi)$ vs $\ln (\ell)$ but the resulting plots were quite noticably curved. Figure 2 shows the dependence of $\ln (\xi)$ on $\ln [\ln (\ell)]$."


## History of the Problem

- Diaconis-Fulton 1991: Addition operation on subsets of $\mathbb{Z}^{d}$.
- Lawler-Bramson-Griffeath 1992: w.p.1,

$$
B_{(1-\varepsilon) r} \subset A\left(\pi r^{2}\right) \subset B_{(1+\varepsilon) r} \quad \text { eventually. }
$$

- Lawler 1995: w.p.1,

$$
\mathbf{B}_{r-r^{1 / 3} \log ^{2} r} \subset A\left(\pi r^{2}\right) \subset \mathbf{B}_{r+r^{1 / 3}} \log ^{4} r \quad \text { eventually. }
$$

"A more interesting question... is whether the errors are $o\left(n^{\alpha}\right)$ for some $\alpha<1 / 3$."

## Logarithmic Fluctuations Theorem

Jerison - L. - Sheffield 2010: with probability 1,

$$
\mathbf{B}_{r-C \log r} \subset A\left(\pi r^{2}\right) \subset \mathbf{B}_{r+C \log r} \text { eventually. }
$$

Asselah - Gaudillière 2010 independently obtained

$$
\mathbf{B}_{r-C \log r} \subset A\left(\pi r^{2}\right) \subset \mathbf{B}_{r+C \log ^{2} r} \text { eventually. }
$$

## Logarithmic Fluctuations in Higher Dimensions

In dimension $d \geq 3$, let $\omega_{d}$ be the volume of the unit ball in $\mathbb{R}^{d}$. Then with probability 1 ,

$$
\mathbf{B}_{r-C \sqrt{\log r}} \subset A\left(\omega_{d} r^{d}\right) \subset \mathbf{B}_{r+C \sqrt{\log r}} \text { eventually }
$$

for a constant $C$ depending only on $d$.
(Jerison - L. - Sheffield 2010; Asselah - Gaudillière 2010)

## Elements of the proof

- Thin tentacles are unlikely.
- Martingales to detect fluctuations from circularity.
- "Self-improvement"


## Thin tentacles are unlikely



A thin tentacle.
Lemma. If $0 \notin \mathbf{B}(z, m)$, then

$$
\mathbb{P}\left\{z \in A(n), \#(A(n) \cap \mathbf{B}(z, m)) \leq b m^{d}\right\} \leq \begin{cases}C e^{-c m^{2} / \log m}, & d=2 \\ C e^{-c m^{2}}, & d \geq 3\end{cases}
$$

for constants $b, c, C>0$ depending only on the dimension $d$.

## Early and late points in $A(n)$, for $n=\pi r^{2}$



## Early and late points

Definition 1. $z$ is an m-early point if:

$$
z \in A(n), \quad n<\pi(|z|-m)^{2}
$$

Definition 2. $z$ is an $\ell$-late point if:

$$
z \notin A(n), \quad n>\pi(|z|+\ell)^{2}
$$

$\mathcal{E}_{m}[n]=$ event that some point in $A(n)$ is $m$-early

$$
\mathcal{L}_{\ell}[n]=\text { event that some point in } \mathbf{B}_{\sqrt{n} / \pi-\ell} \text { is } \ell \text {-late }
$$

## Structure of the argument: Self-improvement

LEMMA 1. No $\ell$-late points implies no m-early points:
If $m \geq C \ell$, then

$$
\mathbb{P}\left(\mathcal{E}_{m}[n] \cap \mathcal{L}_{\ell}[n]^{c}\right)<n^{-10} .
$$

LEMMA 2. No $m$-early points implies no $\ell$-late points:
If $\ell \geq \sqrt{C(\log n) m}$, then

$$
\mathbb{P}\left(\mathcal{L}_{\ell}[n] \cap \mathcal{E}_{m}[n]^{c}\right)<n^{-10} .
$$

Iterate, $\ell \mapsto \sqrt{C(\log n) C \ell}$, which is decreasing until

$$
\ell=C^{2} \log n
$$

## Iterating Lemmas 1 and 2




- Fix $n$ and let $\ell, m$ be the maximal lateness and earliness occurring by time $n$. Iterate starting from $m_{0}=n$ :
- $(\ell, m)$ unlikely to belong to a vertical rectangle by Lemma 1.
- $(\ell, m)$ unlikely to belong to a horizontal rectangle by Lemma 2.


## Early and late point detector

To detect early points near $\zeta \in \mathbb{Z}^{2}$, we use the martingale

$$
M_{\zeta}(n)=\sum_{z \in \widetilde{A}(n)}\left(H_{\zeta}(z)-H_{\zeta}(0)\right)
$$

where $H_{\zeta}$ is a discrete harmonic function approximating $\operatorname{Re}\left(\frac{\zeta /|\zeta|}{\zeta-z}\right)$.


The fine print:

- Discrete harmonicity fails at three points $z=\zeta, \zeta+1, \zeta+1+i$.
- Modified growth process $\widetilde{A}(n)$ stops at $\partial B_{|\zeta|}(0)$.


## Time change of Brownian motion

- To get a continuous time martingale, we use Brownian motions on the grid $\mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z}$ instead of random walks.
- Then there is a standard Brownian motion $B_{\zeta}$ such that

$$
M_{\zeta}(t)=B_{\zeta}\left(s_{\zeta}(t)\right)
$$

where

$$
s_{\zeta}(t)=\lim \sum_{i=1}^{N}\left(M\left(t_{i}\right)-M\left(t_{i-1}\right)\right)^{2}
$$

is the quadratic variation of $M_{\zeta}$.

## LEMMA 1. No $\ell$-late implies no $m=C \ell$-early

Event $Q[z, k]$ :

- $z \in A(k) \backslash A(k-1)$.
- $z$ is m-early $\left(z \in A\left(\pi r^{2}\right)\right.$ for $\left.r=|z|-m\right)$.
- $\mathcal{E}_{m}[k-1]^{c}$ : No previous point is m-early.
- $\mathcal{L}_{\ell}[n]^{c}$ : No point is $\ell$-late.

We will use $M_{\zeta}$ for $\zeta=(1+4 m / r) z$ to show for $0<k \leq n$,

$$
\mathbb{P}(Q[z, k])<n^{-20}
$$

## Main idea: Early but no late would be a large deviation!

- Recall there is a Brownian motion $B_{\zeta}$ such that

$$
M_{\zeta}(n)=B_{\zeta}\left(s_{\zeta}(n)\right) .
$$

- On the event $Q[z, k]$

$$
\begin{equation*}
\mathbb{P}\left(M_{\zeta}(k)>c_{0} m\right)>1-n^{-20} \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbb{P}\left(s_{\zeta}(k)<100 \log n\right)>1-n^{-20} . \tag{2}
\end{equation*}
$$

- On the other hand, $(s=100 \log n)$

$$
\mathbb{P}\left(\sup _{s^{\prime} \in[0, s]} B_{\zeta}\left(s^{\prime}\right) \geq s\right) \leq e^{-s / 2}=n^{-50}
$$

## Proof of (1)

On the event $Q[z, k]$

$$
\mathbb{P}\left(M_{\zeta}(k)>c_{0} m\right)>1-n^{-20}
$$

- Since $z \in A(k)$ and thin tentacles are unlikely, we have with high probability,

$$
\#(A(k) \cap B(z, m)) \geq b m^{2}
$$

- For each of these $b m^{2}$ points, the value of $H_{\zeta}$ is order $1 / m$, so their total contribution to $M_{\zeta}(k)$ is order $m$.
- No $\ell$-late points means that points elsewhere cannot compensate.


## Proof of (2): Controlling the Quadratic Variation

On the event $Q[z, k]$

$$
\mathbb{P}\left(s_{\zeta}(k)<100 \log n\right)>1-n^{-20} .
$$

- Lemma: There are independent standard Brownian motions $B^{1}, B^{2}, \ldots$ such that

$$
s_{\zeta}(i+1)-s_{\zeta}(i) \leq \tau_{i}
$$

where $\tau_{i}$ is the first exit time of $B^{i}$ from the interval $\left(a_{i}, b_{i}\right)$.

$$
\begin{aligned}
a_{i} & =\min _{z \in \partial \tilde{A}(i)} H_{\zeta}(z)-H_{\zeta}(0) \\
b_{i} & =\max _{z \in \partial \tilde{A}(i)} H_{\zeta}(z)-H_{\zeta}(0) .
\end{aligned}
$$

## Proof of (2): Controlling the Quadratic Variation

On the event $Q[z, k]$

$$
\mathbb{P}\left(s_{\zeta}(k)<100 \log n\right)>1-n^{-20}
$$

- By independence of the $\tau_{i}$,

$$
\mathbb{E} e^{s_{\zeta}(k)} \leq \mathbb{E} e^{\left(\tau_{1}+\cdots+\tau_{k}\right)}=\left(\mathbb{E} e^{\tau_{1}}\right) \cdots\left(\mathbb{E} e^{\tau_{k}}\right) .
$$

- By large deviations for Brownian exit times,

$$
\mathbb{E} e^{\tau(-a, b)} \leq 1+10 a b
$$

- Easy to estimate $a_{i}$, and use the fact that no previous point is $m$-early to bound $b_{i}$. Conclude that

$$
\mathbb{E}\left[e^{s_{\zeta}(k)} 1_{Q}\right] \leq n^{50}
$$

## What changes in higher dimensions?

- In dimension $d \geq 3$ the quadratic variation $s_{\zeta}(n)$ is constant order instead of $\log n$.
- So the fluctuations are instead dominated by thin tentacles, which can grow to length $\sqrt{\log n}$.
- Still open: prove matching lower bounds on the fluctuations of order $\log n$ in dimension 2 and $\sqrt{\log n}$ in dimensions $d \geq 3$.


## Part 2: Limiting Shapes

## Internal DLA with Multiple Sources

- Finite set of points $x_{1}, \ldots, x_{k} \in \mathbb{Z}^{d}$.
- Start with $m$ particles at each site $x_{i}$.
- Each particle performs simple random walk in $\mathbb{Z}^{d}$ until reaching an unoccupied site.
- Get a random set of $k m$ occupied sites in $\mathbb{Z}^{d}$.
- The distribution of this random set does not depend on the order of the walks (Diaconis-Fulton '91).


## Questions

- Fix sources $x_{1}, \ldots, x_{k} \in \mathbb{R}^{d}$.
- Run internal DLA on $\frac{1}{n} \mathbb{Z}^{d}$ with $n^{d}$ particles per source.
- As the lattice spacing goes to zero, is there a scaling limit?
- If so, can we describe the limiting shape?
- Recall from part 1: If $k=1$, then the limiting shape is a ball in $\mathbb{R}^{d}$. (Lawler-Bramson-Griffeath '92)

Two-source internal DLA cluster built from overlapping single-source clusters.

## Diaconis-Fulton Addition

- Finite sets $A, B \subset \mathbb{Z}^{d}$.
- In our application, $A$ and $B$ will be overlapping internal DLA clusters from two different sources.
- Write $A \cap B=\left\{y_{1}, \ldots, y_{k}\right\}$.
- To form $A+B$, let $C_{0}=A \cup B$ and

$$
C_{j}=C_{j-1} \cup\left\{z_{j}\right\}
$$

where $z_{j}$ is the endpoint of a simple random walk started at $y_{j}$ and stopped on exiting $C_{j-1}$.

- Define $A+B=C_{k}$.
- Abeilan property: the law of $A+B$ does not depend on the ordering of $y_{1}, \ldots, y_{k}$.



# Diaconis-Fulton sum of two squares in $\mathbb{Z}^{2}$ overlapping in a smaller square. 

## Divisible Sandpile

- Given $A, B \subset \mathbb{Z}^{d}$, start with
- 2 units of mass on each site in $A \cap B$; and
- 1 unit of mass on each site in $A \cup B-A \cap B$.
- At each time step, choose $x \in \mathbb{Z}^{d}$ with mass $m(x)>1$, and distribute the excess mass $m(x)-1$ equally among the $2 d$ neighbors of $x$.
- As $t \rightarrow \infty$, get a limiting region $A \oplus B \subset \mathbb{Z}^{d}$ of sites with mass 1 .
- Sites in $\partial(A \oplus B)$ have fractional mass.
- Sites outside have zero mass.
- Abelian property: $A \oplus B$ does not depend on the choices.



## Divisible sandpile sum of two squares in $\mathbb{Z}^{2}$ overlapping in a smaller square.



Diaconis-Fulton sum


Divisible sandpile sum

## The Odometer Function

- $u(x)=$ total mass emitted from $x$. (gross, not net)
- Discrete Laplacian:

$$
\begin{aligned}
\Delta u(x) & =\frac{1}{2 d} \sum_{y \sim x} u(y)-u(x) \\
& =\text { mass received }- \text { mass emitted } \\
& =1-1_{A}(x)-1_{B}(x), \quad x \in A \oplus B .
\end{aligned}
$$

- Boundary condition: $u=0$ on $\partial(A \oplus B)$.
- Need additional information to determine the domain $A \oplus B$.


## Free Boundary Problem

- Unknown function $u$, unknown domain $D=\{u>0\}$.

$$
\begin{aligned}
& u \geq 0 \\
& \Delta u \leq 1-1_{A}-1_{B} \\
& u\left(\Delta u-1+1_{A}+1_{B}\right)=0 .
\end{aligned}
$$

## The Obstacle Problem

- Given $A, B \subset \mathbb{Z}^{d}$, we define the "obstacle:"

$$
\gamma(x)=-|x|^{2}-\sum_{y \in A} g(x, y)-\sum_{y \in B} g(x, y),
$$

where $g$ is the Green function for simple random walk

$$
g(x, y)=\mathbb{E}_{x} \#\left\{k \mid X_{k}=y\right\}
$$

( $\ln \mathbb{Z}^{2}$, we use the negative of the potential kernel instead.)

- Let $s(x)=\inf \left\{\phi(x) \mid \phi\right.$ is superharmonic on $\mathbb{Z}^{d}$ and $\left.\phi \geq \gamma\right\}$.
- Then the odometer function $=s-\gamma$.
- Obstacle for two overlapping disks $A$ and $B$ :

- Obstacle for two point sources $x_{1}$ and $x_{2}$ :



## The Smash Sum of Two Domains in $\mathbb{R}^{d}$

- $A, B \subset \mathbb{R}^{d}$ bounded open sets such that $\partial A, \partial B$ have zero $d$-dimensional Lebesgue measure.
- We define their smash sum $A \oplus B$ to be the domain

$$
A \oplus B=A \cup B \cup\{s>\gamma\}
$$

where

$$
\gamma(x)=-|x|^{2}-\int_{A} g(x, y) d y-\int_{B} g(x, y) d y
$$

and
$s(x)=\inf \{\phi(x) \mid \phi$ is continuous, superharmonic, and $\phi \geq \gamma\}$.


The smash sum

$$
A \oplus B=A \cup B \cup\{s>\gamma\}
$$

of two overlapping disks $A, B \subset \mathbb{R}^{2}$.

## Properties of the Smash Sum

- $A \cup B \subset A \oplus B$.
- Associativity: $(A \oplus B) \oplus C=A \oplus(B \oplus C)$.
- Volume conservation: $\operatorname{vol}(A \oplus B)=\operatorname{vol}(A)+\operatorname{vol}(B)$.
- Quadrature identity: If $h$ is an integrable superharmonic function on $A \oplus B$, then

$$
\int_{A \oplus B} h(x) d x \leq \int_{A} h(x) d x+\int_{B} h(x) d x .
$$

## Scaling Limit of the Discrete Models

- Let $A, B \subset \mathbb{R}^{d}$ be bounded open sets such that $\partial A, \partial B$ have measure zero.
- Theorem (L.-Peres) With probability one

$$
D_{n}, R_{n}, I_{n} \rightarrow A \oplus B \quad \text { as } n \rightarrow \infty,
$$

where

- $D_{n}, R_{n}, I_{n}$ are the smash sums of $A \cap \frac{1}{n} \mathbb{Z}^{d}$ and $B \cap \frac{1}{n} \mathbb{Z}^{d}$, computed using divisible sandpile, rotor-router, and Diaconis-Fulton dynamics, respectively.
- Convergence is in the sense of $\varepsilon$-neighborhoods: for all $\varepsilon>0$

$$
(A \oplus B)_{\varepsilon}^{: \ddot{\varepsilon} \subset D_{n}, R_{n}, I_{n} \subset(A \oplus B)^{\varepsilon::} \quad \text { for all sufficiently large } n . ~}
$$



Internal DLA


Divisible Sandpile


Rotor-Router Model

## Part 3: Integrality wreaks havoc

## The Abelian Sandpile as a Growth Model

- Start with a pile of $n$ chips at the origin in $\mathbb{Z}^{d}$.
- Each site $x=\left(x_{1}, \ldots, x_{d}\right) \in \mathbb{Z}^{d}$ has $2 d$ neighbors

$$
x \pm e_{i}, \quad i=1, \ldots, d
$$

- Any site with at least $2 d$ chips is unstable, and topples by sending one chip to each neighbor.
- This may create further unstable sites, which also topple.
- Continue until there are no more unstable sites.

Toppling to Stabilize A Sandpile

- Example: $n=16$ chips in $\mathbb{Z}^{2}$.
- Sites with 4 or more chips are unstable.



## Stable.

## Abelian Property

- The final stable configuration does not depend on the order of topplings.
- Neither does the number of times a given vertex topples.


## Sandpile of $1,000,000$ chips in $\mathbb{Z}^{2}$



## Growth on a General Background

- Let each site $x \in \mathbb{Z}^{d}$ start with $\sigma(x)$ chips. $(\sigma(x) \leq 2 d-1)$
- We call $\sigma$ the background configuration.
- Place $n$ additional chips at the origin.
- Let $S_{n, \sigma}$ be the set of sites that topple.


## Constant Background $\sigma \equiv h$



$$
h=2
$$



$$
h=1
$$



$$
h=0
$$

## What about background $h=3$ ?

$$
\begin{aligned}
& \begin{array}{|l|l|l|l|l|l|l|}
\hline 3 & 3 & 5 & 0 & 5 & 3 & 3 \\
\hline \hline 3 & 5 & 1 & 4 & 1 & 5 & 3 \\
\hline \hline & 1 & 5 & 0 & 5 & 1 & 5 \\
\hline 0 & 4 & 0 & 4 & 0 & 4 & \\
\hline 0 & 4 & 0 & 4 & 0 & 4 & 0 \\
\hline 5 & 1 & 5 & 0 & 5 & 1 & 5 \\
\hline 3 & 5 & 1 & 4 & 1 & 5 & 3 \\
\hline 3 & 3 & 5 & 0 & 5 & 3 & 3 \\
\hline
\end{array} \\
& \text {... Never stops toppling! }
\end{aligned}
$$

## The Odometer Function

- $u(x)=$ number of times $x$ topples.
- Discrete Laplacian:

$$
\begin{aligned}
\Delta u(x) & =\sum_{y \sim x} u(y)-2 d u(x) \\
& =\text { chips received }- \text { chips emitted } \\
& =\tau^{\circ}(x)-\tau(x)
\end{aligned}
$$

where $\tau$ is the initial unstable chip configuration and $\tau^{\circ}$ is the final stable configuration.

## Stabilizing Functions

- Given a chip configuration $\tau$ on $\mathbb{Z}^{d}$ and a function $u_{1}: \mathbb{Z}^{d} \rightarrow \mathbb{Z}$, call $u_{1}$ stabilizing for $\tau$ if

$$
\tau+\Delta u_{1} \leq 2 d-1 .
$$

- If $u_{1}$ and $u_{2}$ are stabilizing for $\tau$, then

$$
\begin{aligned}
\tau+\Delta \min \left(u_{1}, u_{2}\right) & \leq \tau+\max \left(\Delta u_{1}, \Delta u_{2}\right) \\
& =\max \left(\tau+\Delta u_{1}, \tau+\Delta u_{2}\right) \\
& \leq 2 d-1
\end{aligned}
$$

so $\min \left(u_{1}, u_{2}\right)$ is also stabilizing for $\tau$.

## Least Action Principle

- Let $\tau$ be a chip configuration on $\mathbb{Z}^{d}$ that stabilizes after finitely many topplings, and let $u$ be its odometer function.
- Least Action Principle:

$$
\text { If } u_{1}: \mathbb{Z}^{d} \rightarrow \mathbb{Z}_{\geq 0} \text { is stabilizing for } \tau \text {, then } u \leq u_{1}
$$

- So the odometer is minimal among all nonnegative stabilizing functions:

$$
u(x)=\min \left\{u_{1}(x) \mid u_{1} \geq 0 \text { is stabilizing for } \tau\right\} .
$$

- Interpretation: "Sandpiles are lazy."


## Obstacle Problem with an Integrality Condition

- Lemma. The abelian sandpile odometer function is given by

$$
u=s-\gamma
$$

where

$$
s(x)=\min \left\{\begin{array}{l|l}
f(x) & \begin{array}{c}
f: \mathbb{Z}^{d} \rightarrow \mathbb{R} \text { is superharmonic } \\
\text { and } f-\gamma \text { is } \mathbb{Z}_{\geq 0} \text {-valued }
\end{array}
\end{array}\right\}
$$

- The obstacle $\gamma$ is given by

$$
\gamma(x)=-\frac{(2 d-1)|x|^{2}+n \cdot g(o, x)}{2 d}
$$

where $g$ is the Green's function for simple random walk in $\mathbb{Z}^{d}$

$$
g(o, x)=\mathbb{E}_{o} \#\left\{k \mid X_{k}=x\right\}
$$



Abelian sandpile
(Integrality constraint)


Divisible sandpile
(No integrality constraint)

## Sandpile growth rates

- Let $S_{n, d, h}$ be the set of sites in $\mathbb{Z}^{d}$ that topple, if $n+h$ chips start at the origin and $h$ chips start at every other site in $\mathbb{Z}^{d}$.

Theorem (Fey-L.-Peres) If $h \leq 2 d-2$, then

$$
B_{c n^{1 / d}} \subset S_{n, d, h} \subset B_{C n^{1 / d}} .
$$

- Extends earlier work of Fey-Redig and Le Borgne-Rossin.


## Bounds for the Abelian Sandpile Shape


(Disk of area $n / 3) \subset S_{n} \subset($ Disk of area $n / 2)$

## A Few Extra Chips Produce An Explosion

- Let $(\beta(x))_{x \in \mathbb{Z}^{d}}$ be independent Bernoulli random variables

$$
\beta(x)= \begin{cases}1 & \text { with probability } \varepsilon \\ 0 & \text { with probability } 1-\varepsilon\end{cases}
$$

- Theorem (Fey-L.-Peres) For any $\varepsilon>0$, with probability 1 , the background $2 d-2+\beta$ on $\mathbb{Z}^{d}$ is explosive.
- i.e., for large enough $n$, adding $n$ chips at the origin causes every site in $\mathbb{Z}^{d}$ to topple infinitely many times.
- Same is true if the extra chips start on an arbitrarily sparse lattice $L \subset \mathbb{Z}^{d}$, provided $L$ meets every coordinate plane $\left\{x_{i}=k\right\}$.


## How to Prove An Explosion

- Claim: If every site in $\mathbb{Z}^{d}$ topples at least once, then every site topples infinitely often.
- Otherwise, let $x$ be the first site to finish toppling.
- Each neighbor of $x$ topples at least one more time, so $x$ receives at least $2 d$ additional chips.
- So $x$ must topple again. $\Rightarrow \Leftarrow$


## Straley's Argument for Bootstrap Percolation

- Let $E_{k}$ be the event that each face of the cube $Q_{k}$ starts with at least one extra chip. Then

$$
\mathbb{P}\left(E_{k}^{c}\right) \leq 2 d(1-\varepsilon)^{k}
$$

- By Borel-Cantelli, with probability 1 almost all $E_{k}$ occur.


## An Explosion In Progress



- Sites colored black are unstable. All sites in $\mathbb{Z}^{2}$ will topple infinitely often!


## A Mystery: Scale Invariance

- Big sandpiles look like scaled up small sandpiles!
- Let $\sigma_{n}(x)$ be the final number of chips at $x$ in the sandpile of $n$ particles on $\mathbb{Z}^{d}$.
- Squint your eyes: for $x \in \mathbb{R}^{d}$ let

$$
f_{n}(x)=\frac{1}{a_{n}^{2}} \sum_{\substack{y \in \mathbb{Z}^{d} \\\|y-\sqrt{n} x\| \leq a_{n}}} \sigma_{n}(y) .
$$

where $a_{n}$ is a sequence of integers such that

$$
a_{n} \uparrow \infty \quad \text { and } \quad \frac{a_{n}}{\sqrt{n}} \downarrow 0 .
$$

## Scale Invariance Conjecture

- Conjecture: There is a sequence $a_{n}$ and a function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}_{\geq 0}$ which is locally constant on an open dense set, such that $f_{n} \rightarrow f$ at all continuity points of $f$.
- Now partly proved! Pegden and Smart (arXiv:1105.0111) show existence of a weak-* limit for $f_{n}$ !


## Two Sandpiles of Different Sizes



(scaled down by $\sqrt{2}$ )

## Locally constant "steps" of $f$ correspond to periodic patterns:



## A Mystery: Dimensional Reduction

- Our argument used simple properties of one-dimensional sandpiles to bound the diameter of higher-dimensional sandpiles.
- Deepak Dhar pointed out that there seems to be a deeper relationship between sandpiles in $d$ and $d-1$ dimensions...


## Dimensional Reduction Conjecture

- $\sigma_{n, d}$ : sandpile of $n$ chips on background $h=2 d-2$ in $\mathbb{Z}^{d}$.
- Conjecture: For any $n$ there exists $m$ such that

$$
\sigma_{n, d}\left(x_{1}, \ldots, x_{d-1}, 0\right)=2+\sigma_{m, d-1}\left(x_{1}, \ldots, x_{d-1}\right)
$$

for almost all $x$ sufficiently far from the origin.

## A Two-Dimensional Slice of A Three-Dimensional Sandpile


$d=3$ (slice through origin)

$$
\begin{gathered}
h=4 \\
n=5,000,000
\end{gathered}
$$



$$
\begin{gathered}
d=2 \\
h=2 \\
m=46,490
\end{gathered}
$$

## Thank You!



References:

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