Weak and strong survival for branching random walks on weighted graphs

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Outline
1 BRW on weighted graphs
   • Setting the frame
   • Asymptotic degrees

2 Our main results
   • The local behavior
   • The global behavior

Paper source
Daniela Bertacchi, F.Z.,

- Critical behaviors and critical values of branching random walks on multigraphs,

- Characterization of the critical values of branching random walks on weighted graphs through infinite-type branching processes,
  arXiv:0804.0224

The colleague I share responsibility with

Daniela Bertacchi, F.Z.,

- Critical behaviors and critical values of branching random walks on multigraphs,

- Characterization of the critical values of branching random walks on weighted graphs through infinite-type branching processes,
  arXiv:0804.0224
**Space and weights**

Set: \( X \) (at most countable) set of **vertices**.

Rates: nonnegative matrix \( K = (k_{xy})_{x,y \in X} \),
s.t. \( k(x) := \sum_{y \in X} k_{xy} < +\infty \) for all \( x \in X \).

Edges: \( E(X) := \{(x,y) : k_{xy} > 0\} \)

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Birth rate \( \lambda k_{xy} \)

\( x \) \( y \) \( z \)

**Examples**: Edge breeding vs. Site breeding
- \( k_{xy} \) is the number of bonds from \( x \) to \( y \) in a multigraph
- \( K \) is a transition probability matrix
Weak and strong survival for BRW

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**Outline**

1. **BRW on weighted graphs**
   - Setting the frame
   - Asymptotic degrees

2. **Our main results**
   - The local behavior
   - The global behavior
Asymptotic degrees

Define recursively

\[
\begin{align*}
k^{n+1}_{xy} &= \sum_{z \in X} k^n_{xz} k^0_{zy}, \quad \forall x, y \in X \\
k^0_{xy} &= \delta_{xy}
\end{align*}
\]

- \(M_s := \limsup_n \sqrt[n]{k^n_{xy}}\) strong (or local) degree
- \(M_w := \liminf_n \sqrt[n]{\sum_{y \in X} k^n_{xy}}\) weak (or global) degree

They do not depend on the choice of \(x \in X\) since \((X, E(X))\) is connected.

Clearly \(M_s \leq M_w\)

If \(K\) is the adjacency matrix of a graph this is the number of paths of length \(n\) from \(x\) to \(x\)
Asymptotic degrees

Define recursively
\[
\begin{cases}
k_{xy}^{n+1} &= \sum_{z \in X} k_{xz}^n k_{zy}, \\
k_y^0 &= \delta_{xy},
\end{cases}
\forall x, y \in X
\]

- \( M_s := \limsup_n \sqrt[n]{k_{xx}^n} \) \quad \text{(strong or local degree)}
- \( M_w := \liminf_n \sqrt[n]{\sum_{y \in X} k_{xy}^n} \) \quad \text{(weak or global degree)}

If \( K \) is the adjacency matrix of a graph this is the number of paths of length \( n \) from \( x \) ending anywhere.

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Weak and strong survival for BRW

Outline

1. BRW on weighted graphs
   - Setting the frame
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   Our main results
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   The strong critical value

   Let \( \{\eta_t\} \) be a BRW starting with one particle at some site \( x_0 \).

   strong critical value
   \[\lambda_s := \inf \{ \lambda : \Pr(\eta_t(x) > 0 \text{ for arbitrarily large } t > 0) \} \]

   It does not depend on \( x \in X \) since \((X, E(X))\) is connected.

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Weak and strong survival for BRW
The strong critical value

Let \( \{\eta_t\}_t \) be a BRW starting with one particle at some site \( x_0 \).

**strong critical value**

\[
\lambda_s := \inf \{ \lambda : \mathbb{P}(\eta_t(x) > 0 \text{ for arbitrarily large } t) > 0 \}
\]

**Theorem**

For every weighted graph \((X, K)\) we have that \( \lambda_s = 1/M_s \).

If \( \lambda > 1/M_s \) there is local survival

The colony survives (with positive probability) returning infinitely often in any site.

If \( \lambda < 1/M_s \) there is a.s. local extinction

Do we know what happens if \( \lambda = 1/M_s \)?
Let $\{\eta_t\}_t$ be a BRW starting with one particle at some site $x_0$.

**Strong critical value**

$$\lambda_s := \inf\{\lambda : \mathbb{P}(\eta_t(x) > 0 \text{ for arbitrarily large } t) > 0\}$$

**Theorem**

For every weighted graph $(X,K)$ we have that $\lambda_s = 1/M_s$.

---

Let $\{\eta_t\}_t$ be a realization of the BRW.

**Weak critical value**

$$\lambda_w := \inf\{\lambda : \mathbb{P}(\sum_{x \in X} \eta_t(x) > 0 \text{ for all } t) > 0\}$$

**Theorem**

For every weighted graph $(X,K)$ we have that $\lambda_w = \inf\{\lambda \in \mathbb{R} : \exists v \in [0,1]^X, v > 0, \lambda Kv > v/(1-v)\}$
The weak critical value

Let \( \{ \eta_t \}_{t \geq 0} \) be a realization of the BRW.

**weak critical value**

\[
\lambda_w := \inf \{ \lambda : P(\sum_{x \in X} \eta_t(x) > 0 \text{ for all } t > 0) \}
\]

**Theorem**

Let \((X, K)\) be any weighted graph.

- The \( \lambda \)-BRW survives globally if and only if there exists \( v \in [0,1]^X \), \( v > 0 \) such that \( \lambda K v \geq v/(1 - v) \).
- \( \lambda_w := \inf \{ \lambda \in \mathbb{R} : \exists v \in [0,1]^X, v > 0, \lambda K v \geq v/(1 - v) \} \)

Corollary

For every weighted graph \((X, K)\) we have that \( \lambda_w \geq 1/\inf K \wedge \).

---

The weak critical value

Let \( \{ \eta_t \}_{t \geq 0} \) be a realization of the BRW.

**weak critical value**

\[
\lambda_w := \inf \{ \lambda : P(\sum_{x \in X} \eta_t(x) > 0 \text{ for all } t > 0) \}
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**Theorem**

Let \((X, K)\) be any weighted graph.

- The \( \lambda \)-BRW survives globally if there exists \( v \in [0,1]^X \), \( v > 0 \) such that \( \lambda K v \geq v/(1 - v) \).
- \( \lambda_w := \inf \{ \lambda \in \mathbb{R} : \exists v \in [0,1]^X, v > 0, \lambda K v \geq v/(1 - v) \} \)

**When is there an equality here?**

For every weighted graph \((X, K)\) we have that \( \lambda_w \geq 1/\inf K \wedge \).
We say that \((X, K)\) is an \(\mathcal{F}\)-weighted graph if there exists a weighted graph \((Y, \tilde{K})\) and a surjective map \(f : X \rightarrow Y\) such that
- \(Y\) is finite;
- \(\sum_{z \in f^{-1}(y)} k_{xz} = \tilde{k}_{f(x)y}\) for all \(x \in X\) and \(y \in Y\).

**Meaning of the definition**
Fix a vertex \(x\) and a label \(y\)

We say that \((X, K)\) is an \(\mathcal{F}\)-weighted graph if there exists a weighted graph \((Y, \tilde{K})\) and a surjective map \(f : X \rightarrow Y\) such that
- \(Y\) is finite;
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\begin{align*}
&Y \text{ is finite;} \\
&\sum_{z \in f^{-1}(y)} k_{xz} = \tilde{k}(x) y \quad \text{for all } x \in X \text{ and } y \in Y.
\end{align*}
\]

**Meaning of the definition**
This total rate depends only on the labels \(f(x)\) and \(y\).

\((X, K)\) “inherits” from \((Y, \tilde{K})\) its \(M_w\).

The \(\lambda\)-BRWs on \((X, K)\) and on \((Y, \tilde{K})\) have the same global behavior.

**Examples of BRWs on \(\mathcal{F}\)-weighted graphs**

Edge-breeding BRWs on quasi-transitive graphs are BRWs on \(\mathcal{F}\)-weighted graphs.

Site-breeding BRWs are BRWs on \(\mathcal{F}\)-weighted graphs.
BRW on weighted graphs
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Examples

Weighted graph Y

Graph X

Weighted graph Y

Graph X

Weighted graph Y

Graph X

Weighted graph Y

Graph X

Weighted graph Y

Graph X

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Weak and strong survival for BRW
Examples

Weighted graph Y

Graph X

Theorem

For every $F$-weighted graph $(X,K)$ we have that $\lambda_w = 1/M_w$.}

Theorem

For every $F$-weighted graph $(X,K)$ we have that the (critical) $\lambda_w$-BRW dies out globally a.s.

Example of critical global survival

Let $X := \mathbb{N}$ and $K$ be defined by $k_{0,1} := 2$, $k_{n,n+1} := (1 + 1/n)^2$, $k_{n+1,n+1} := 1/3^{n+1}$ and 0 otherwise.

It is possible to find examples of BRWs on weighted graphs which survive globally with positive probability at the critical value $\lambda_w$. 

Example of critical global survival

Let $X := \mathbb{N}$ and $K$ be defined by $k_{01} := 2$, $k_{n+1} := (1 + 1/n)^2$, $k_{n+1,n} := 1/3^{n+1}$ and 0 otherwise.

(very) quick sketch

- No solutions in $l^\infty(X)$ of $\lambda K v \geq v$ with $v > 0$ if $\lambda < 1$

Note that this is the non linear inequality

$\lambda K v \geq v/(1 - v)$ with $\lambda = 1$

(very) quick sketch

- No solutions in $l^\infty(X)$ of $\lambda K v \geq v$ with $v > 0$ if $\lambda < 1$
- Explicit solution in $[0, 1]^X$ of $K v \geq v/(1 - v)$ with $v > 0$
Let $X := \mathbb{N}$ and $K$ be defined by

\[ k_0 := 2, \quad k_{n+1} := \frac{1 + 1/n^2}{n^{1/3}}, \quad k_{n+2} := \frac{1/3}{n+1} \quad \text{and} \quad 0 \text{ otherwise.} \]

**Example of critical global survival**

Let $X := \mathbb{N}$ and $K$ be defined by

\[ k_0 := 2, \quad k_{n+1} := \left(1 + \frac{1}{n}\right)^2, \quad k_{n+1} := \frac{1}{3n+1} \quad \text{and} \quad 0 \text{ otherwise.} \]

We look now for a pure weak phase of the BRW.

\[
\begin{array}{cccc}
(0, \lambda_w) & (\lambda_w, \lambda_s) & (\lambda_s, +\infty) \\
\text{global extinction} & \text{global survival} & \text{global survival} \\
\text{local extinction} & \text{local extinction} & \text{local survival}
\end{array}
\]

(very) quick sketch

~ No solutions in $l^\infty(X)$ of $\lambda K v \geq v$ with $\nu > 0$ if $\lambda < 1$

~ Explicit solution in $[0, 1]^X$ of $K v \geq \nu/(1 - \nu)$ with $\nu > 0$

\[
\begin{array}{cccc}
\lambda_w > 1 & \text{global survival if} & \lambda = 1, \text{ hence} & \lambda_w \leq 1
\end{array}
\]

We look now for a pure weak phase of the BRW, that is $\lambda_w < \lambda_s$.

\[
\begin{array}{cccc}
(0, \lambda_w) & (\lambda_w, \lambda_s) & (\lambda_s, +\infty) \\
\text{global extinction} & \text{global survival} & \text{global survival} \\
\text{local extinction} & \text{local extinction} & \text{local survival}
\end{array}
\]

**Theorem**

Let $(X, K)$ be a $\mathcal{F}$-weighted graph such that $k_{xy} = k_{yx}$ for all $x, y \in X$. Then $\lambda_w < \lambda_s$ if and only if $(X, K)$ is nonamenable.
Pure weak phase

We look now for a pure weak phase of the BRW, that is $\lambda_w < \lambda_s$.

Theorem

Let $(X, K)$ be a $\mathcal{F}$-weighted graph such that $k_{xy} = k_{yx}$ for all $x, y \in X$. Then $\lambda_w < \lambda_s$ if and only if $(X, K)$ is nonamenable.

In general nonamenability is not equivalent to the existence of a pure weak phase.

Open questions

Is it true that $\lambda_w = 1/M_w$ for every weighted graph?

Is it possible to find an edge-breeding BRW (that is, $K$ with nonnegative integer elements) on a graph which survives globally (with positive probability) at the critical value $\lambda_w$?

Amenable weighted graphs

A weighted graph is nonamenable if

$$\inf \left\{ \frac{\sum_{x \in S, y \notin S} k_{xy}}{|S|} : S \subseteq X, |S| < \infty \right\} =: \iota_X > 0.$$