A spatial population model with random obstacles

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Mild obstacles

Model.

Let $\omega$ be a Poisson point process (PPP) on $\mathbb{R}^d$ with intensity $\nu > 0 \sim \mathbb{P}$. $a > 0$ and $\beta_2 > \beta_1 > 0$ fixed.

$$K = K_\omega := \bigcup_{x \in \text{supp}(\omega)} \bar{B}(x, a) .$$

$K$: Mild obstacle configuration attached to $\omega$.

$K^c$: ‘Swiss cheese’

Given $\omega$, we define $P^\omega$ as the law of the (strictly dyadic) BBM on $\mathbb{R}^d$, $d \geq 1$ with spatially dependent branching rate

$$\beta := \beta_1 1_K + \beta_2 1_{K^c} .$$

The process $Z$ under $P^\omega$ is called a BBM with mild Poissonian obstacles.
Mild obstacles

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Questions

- Growth of the global/local population size? How much will the absence of branching in $K$ slow the global reproduction down? Change the exponent $\beta_2$?
- What are the large deviations? [E.g. $P(\text{atypically small population})=?$]
- How about the spatial spread? How will the $\sqrt{\beta_2}$ speed reduce?

Questions can be asked in 2 diff. ways: annealed and quenched sense.
Related models in biology

(i) **Migration with unfertile areas** (Population dynamics):
Population moves in space and reproduces by binary splitting, but randomly located reproduction-suppressing areas modify the growth.

(ii) **Fecundity selection** (Genetics):
Reproduction and mutation. Certain randomly distributed genetic types have low fitness: even though they can be obtained by mutation, they themselves do not reproduce easily, unless mutation transforms them to different genetic types.

['Space' = space of genetic types rather than physical space.]

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- **Population dynamics setting**: Is there a preferred spatial location for the process to populate?
- **Genetic setting**: Existence of a certain kind of genetic type that is preferred in the long run that lowers the risk of low of fecundity caused by mutating into less fit genetics types?
Genealogical structure — exciting problem!
E.g. it seems quite possible that for large times the ‘bulk’ of the population consists of descendants of a single particle that

- decided to travel far enough (resp. to mutate many times)
- reached a less hostile environment (resp. in high fitness genetic type area), where she and her descendants can reproduce freely.

Related phenomenon in marine systems: hypoxic patches form in estuaries because of stratification of the water.
⇒ The patches affect different organisms in different ways but are detrimental to some of them. They appear and disappear in an effectively stochastic way.

Source-sink theory: some patches of habitat are good for a species (and growth rate is positive) whereas other patches are poor (and growth rate smaller, or is zero or negative). Individuals can move between patches randomly or according to more detailed biological rules for behavior.
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Systems with periodic local disturbances like e.g.

- FORESTS where trees sometimes fall creating gaps (which have various effects on different species but may harm some)
- AREAS OF GRASS or brush which are subject to occasional fires — burned areas can be expected to less suitable habitats for at least some organisms.

Result on population size

Define average growth rate:

\[ r_t(\omega) := \frac{\log |Z_t(\omega)|}{t}. \]

Theorem

On a set of full $P$-measure,

\[ \lim_{t \to \infty} (\log t)^{2/d}(r_t - \beta_2) = -c(d, \nu), \]

in $P^\omega$-probability.

That is, loosely speaking,

\[ r_t \approx \beta_2 - c(d, \nu)(\log t)^{-2/d}. \]
**Result on spatial spread**

An upper estimate on the spatial spread. The order of the correction term is larger than the \( O(\log t) \) term in a result of Bramson, namely it is

\[
O\left(\frac{t}{(\log t)^{2/d}}\right).
\]

**Theorem**

\( \beta_1 \) plays no role and, loosely speaking, at time \( t \) the spread of the process

\[
\leq t\sqrt{2\beta_2} - c(d,\nu)\sqrt{\frac{\beta_2}{2}} \cdot \frac{t}{(\log t)^{2/d}}.
\]

**Problem (Shape of branching tree)**

How does the discrete probability measure valued process

\[
\tilde{Z}_t(\cdot) := \frac{Z_t(\cdot)}{|Z_t|}
\]

look like? Is it true that \( \exists \) Unique dominant branch?

**Problem (More general branching)**

What happens when dyadic branching is replaced by a general one?

E.g. critical branching. Let \( \beta_1 = 0 \). It is still true (nontrivial) that

\[
P^\omega(\text{extinction}) = 1, \; a.e. \; \omega \in \Omega.
\]

**Q:** What is the order of the tail \( P(\tau_{\text{ext}} > t) \)?

**THANK YOU FOR YOUR ATTENTION!**