BASIC TOPICS IN HARMONIC ANALYSIS II: SINGULAR INTEGRALS

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Abstract. These notes are the second part to [DE00]. Here we discuss the Calderón–Zigmund decomposition and its application to the theory of singular integrals.

1. Introduction

As in [DE00], this set of notes touches classical topics in harmonic analysis. The emphasis is on the important technique, known as the Claderón–Zigmund decomposition. As an application, the theory of singular integrals is develop.

The presentation is self–contained and follows the presentation in [Ste70]. We expect to add more topics in the future.

2. Whitney Decomposition in $\mathbb{R}^n$

We open this section with a couple of theorems concerning separation of open and closed sets in $\mathbb{R}^n$.

In what follows, F will denote an arbitrary non-empty closed set in $\mathbb{R}^n$, Ω its complement. By a cube we mean a closed cube in $\mathbb{R}^n$, with sides parallel to the axis, and two such cubes will be said to be *disjoint* if their interiors are disjoint. For such a cube Q, $\operatorname{diam} Q$ denotes its diameter, and $d(Q,F)$ its distance from F.

Theorem 2.1. *Let F be given. Then there exists a collection af cubes $\mathcal{F}$, $\mathcal{F} = \{Q_1, ...Q_k, ...\}$ so that*

1. $\bigcup_{k=1}^{\infty} Q_k = \Omega$,
2. *The Q_k are mutually* disjoint,
3. $\operatorname{diam} Q_k \leq d(Q_k, F) \leq 4\ \operatorname{diam} Q_k$

Proof. Consider the lattice of points in $\mathbb{R}^n$ whose coordinates are integers. This lattice determines a mesh $\mathcal{M}_0$, which is a collection of cubes. The mesh $\mathcal{M}_0$ leads to a two-way infinite chain of such meshes $\{\mathcal{M}_k\}_{k=-\infty}^{k=\infty}$ with $\mathcal{M}_k = 2^{-k}\mathcal{M}_0$.

Thus each cube in the mesh $\mathcal{M}_k$ gives rise to 2^n cubes in the mesh $\mathcal{M}^{k+1}$ by bisecting the sides. In addition to the meshes $\mathcal{M}_k$, we consider the layers Ω_k defined by $\Omega_k = \{x :\ c\ 2^{-k} < d(x,F) \leq c\ 2^{-k+1}\}$, where c is a positive constant we shall fix momentarily. Obviously $\Omega = \bigcup_k \Omega_k$.

Now we make an initial choice of cubes, which we will denote by $\mathcal{F}_0$, as follows:

$$\mathcal{F}_0 = \bigcup_k \{Q \in \mathcal{M}_k :\ Q \cap \Omega_k \neq \emptyset\}$$

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By setting $c = 2\sqrt{n}$ and taking $Q \in \mathcal{F}_0$, we get the following results:

$$\bigcup_{Q \in \mathcal{F}_0} Q = \Omega$$

and

$$\operatorname{diam} Q \leq d(F,Q) \leq 4 \ \operatorname{diam} Q$$

Only the latter requieres a proof: let $Q \in \mathcal{M}_k$ with $Q \cap \Omega_k \neq \emptyset$. Then, $\operatorname{diam} Q = 2^{-k} \ \sqrt{n}$. For any $x \in Q \cap \Omega_k$, we have

$$d(Q,F) \leq d(x,F) \leq c \ 2^{-k+1} = (2\, c/\sqrt{n}) \ \operatorname{diam} Q,$$

and

$$d(Q,F) \geq d(x,F) - \operatorname{diam} Q > c \ 2^{-k} - \sqrt{n} \ 2^{-k} = (c/\sqrt{n}) \ \operatorname{diam} Q - \operatorname{diam} Q$$

Our choice of c gives the desired results.

We refine now our choice of cubes to get the additional property of disjoint interiors. We achieve this by noting first that any two cubes in $\mathcal{F}_0$ have either disjoint interiors or one is contained in the other. Secondly, for any cube $Q' \in \mathcal{F}_0$, if $Q \in \mathcal{F}_0$ contains Q' then $\operatorname{diam} Q \leq 4 \operatorname{diam} Q'$. This shows that there is unique maximal cube Q that contains Q'. Let $\mathcal{F}$ be the collection of all maximal cubes in $\mathcal{F}_0$.Then we have that:

(1) $\bigcup_{Q \in \mathcal{F}} Q = \Omega$,
(2) the cubes of $\mathcal{F}$ have disjoint interiors from one another,
(3) $\operatorname{diam} Q \leq d(Q,F) \leq d(Q',F) \leq 4 \ \operatorname{diam} Q' \leq 4 \operatorname{diam} Q$, whenever $Q' \in \mathcal{F}_0$ and $Q' \subset Q$.

This ends the proof of the theorem. □

We shall make some observations about the family $\mathcal{F}$ of cubes, whose existence is warranteed by theorem 1, as propositions.

Proposition 2.2. *Suppose Q_1 and Q_2 touch. Then,*

$$\frac{1}{4} \ \operatorname{diam} Q_2 \leq \operatorname{diam} Q_1 \leq 4 \ \operatorname{diam} Q_2$$

Proof. Let $z \in Q_1 \cap Q_2$, then for any $q_1 \in Q_1$ and for any $x \in F$, we have

$$\begin{aligned} d(Q_2,F) \leq d(x,z) &\leq d(z,q_1) + d(q_1,x) \\ &\leq \operatorname{diam} Q_1 + d(q_1,x) \end{aligned}$$

Taking inf on the last inequality produces

$$\begin{aligned} d(Q_2,F) &\leq \operatorname{diam} Q_1 + d(Q_1,F) \\ &\leq \operatorname{diam} Q_1 + 4 \ \operatorname{diam} Q_1 \\ &= 5 \ \operatorname{diam} Q_1 \end{aligned}$$

Since $\operatorname{diam} Q_2 \leq d(Q_2,F)$, it follows that $\operatorname{diam} Q_2 \leq 5 \ \operatorname{diam} Q_1$. However, $\operatorname{diam} Q_2 = 2^k \ \operatorname{diam} Q_1$ for some integer k, thus $k \leq 2$. Therefore $\operatorname{diam} Q_2 \leq 4 \ \operatorname{diam} Q_1$ The previous argument is symetrical on Q_1 and Q_2. □

Proposition 2.3. *Suppose $Q \in \mathcal{F}$, then there are at most $N = 12^n$ cubes in $\mathcal{F}$ which touch Q.*

Proof. If the cube $Q \in \mathcal{F}$ belongs to the mesh $\mathcal{M}_k$, then as it is easily seen, there are 3^n cubes (including Q) which belong to the mesh $\mathcal{M}_k$ and touch Q. Next, each cube in the mesh $\mathcal{M}_k$ can contain at most 4^n cubes of $\mathcal{F}$ of diameter $\geq (1/4)\ \operatorname{diam} Q$. If we combine this with proposition 1, we get the proof of proposition 2. $\square$

Let now Q_k denote any cube in $\mathcal{F}$. Write x^k as the center of this cube and l_k as the common length of its sides. Then of course $\operatorname{diam} Q = \sqrt{n}\, l_k$. For any ε, $0 < \varepsilon < 1/4$, which is arbitrary but will be kept fixed for what follows, denote by Q_k^* the cube which has the same center as Q_k but is expanded by a factor $1+\varepsilon$; that is $Q_k^* = (1+\varepsilon)\left[Q_k - x^k\right] + x^k$. Clearly $Q_k \subset Q_k^*$, and the cubes Q_k^* no longer have disjoint interiors. However the following holds:

Proposition 2.4. *Each point of Ω is contained in at most N of the cubes Q_k^*.*

Proof. Let Q and Q_k be two cubes of $\mathcal{F}$. We claim that $Q_k^* \cap Q \neq \emptyset$ only if $Q_k \cap Q \neq \emptyset$. Consider the union of all cubes in $\mathcal{F}$ which touch Q_k. Since the diameters of these cubes are all $\geq (1/4) \operatorname{diam} Q_k$ by proposition 1, it is clear that this union contains Q_k^*. If $Q_k^* \cap Q \neq \emptyset$, then Q must touch Q_k. However, any point $x \in \Omega$ belongs to some cube Q, therefore by proposition 2 there are at most N cubes Q_k^* which contain x. $\square$

The previous observations allow us to construct the following partition of the unity. Let Q_0 denote the cube of unit length centered at the origin. fix a C^∞ funtion ϕ with properties: $0 \leq \phi \leq 1$; $\phi(x) = 1, x \in Q_0$; and $\phi(x) = 0, x \notin (1+\varepsilon)\ Q_0$.
Let ϕ_k denote the function ϕ adjusted to the cube Q_k; that is

$$\phi(x) = \phi\left(\frac{x - x_k}{l_k}\right).$$

We now defined $\phi_k^*(x)$ for $x \in \Omega$ by

$$\phi_k^*(x) = \frac{\phi_k(x)}{\Phi(x)},$$

where $\Phi(x) = \sum_k \phi_k(x)$. Proposition 3 warranties the identity

$$\sum_k \phi_k^*(x) = 1, \qquad x \in \Omega$$

then defines a partition of the unity.

One application of Whitney decomposition is to the theory of maximal funtion. We present the following theorem:

Theorem 2.5. *Let f be a non-negative integrable function on $\mathbb{R}^n$ and α be a positive constant. Then there exists a decomposition of $\mathbb{R}^n$ so that*

- *i): $\mathbb{R}^n = F \cup \Omega,\ F \cap \Omega = \emptyset$*
- *ii): $f(x) \leq \alpha$ a.e. on F*
- *iii): Ω is the union of cubes, $\Omega = \bigcup_k Q_k$ whose interiors are disjoint and so that for each Q_k*

$$\frac{1}{m(Q_k)} \int_{Q_k} f\, dx \leq B\,\alpha$$

- *iv): $m(\Omega) \leq \frac{A}{\alpha}\ \|f\|_1$*

where A and B are constant that depend only on dimension n of the space

Proof. Let $\Omega = \{M\,f > \alpha\}$, where $M\,f$ is the Hardy maximal function of f which is defined as

$$M\,f(x) = \sup_{r>0} \frac{1}{m\,(B_r(x))} \int_{B_r(x)} f(y)\,dy$$

Then Ω is an open set. By the Maximal function theorem [Rud87, Chap. 7], we have

$$m(\Omega) \leq \frac{3^n}{\alpha} \|f\|_1$$

Therefore, for almost every point $x \in F = \mathbb{R}^n \setminus \Omega$, the following holds true:

$$f(x) = \lim_{r\to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} f \leq M\,f(x) \leq \alpha$$

By theorem 2.1, there is a countable collection of cubes Q_k such that $\Omega = \bigcup_k Q_k$ and $\operatorname{diam} Q_k \leq d(Q_k, F) \leq 4 \operatorname{diam} Q_k$. Let Q_k be one of these cubes and $p_k \in F$ such that $d(F, Q_k) = d(p_k, Q_k)$. Let B_k be the smallest ball whose center is p_k and contains the interior of Q_k. Let be $\gamma_k = m(B_k)/m(Q_k)$. Since $p_k \in F$, we have

$$\alpha \geq\; M\,f(p_k) \geq \frac{1}{m(B_k)} \int_{B_k} f \geq \frac{1}{\gamma_k\, m(Q_k)} \int_{Q_k} f \tag{1}$$

There are points q_k and $q\prime_k$ in Q_k such that $r_k = \operatorname{diam} B_k/2 = d(p_k, q_k)$ and $d(F, Q_k) = d(p_k, q\prime_k)$. Basic geometric arguments and theorem 1 lead to

$$\begin{aligned}
\gamma_k &= \frac{m(B_k)}{m(Q_k)} \\
&= \frac{r_k^n\, m(B_1(0))\ (\sqrt{n})^n}{diam(Q_k)} \\
&\leq \frac{(d(F, Q_k) + \operatorname{diam} Q_k)^n}{(\operatorname{diam} Q_k)^n} \left(\sqrt{n}\right)^n\ m(B_1(0)) \\
&\leq 5^n\ \left(\sqrt{n}\right)^n\ m(B_1(0)) := B
\end{aligned}$$

Finally, substitution of the last inequality into equation (1) gives *iii)*. ☐

3. Basic Interpolation Theorems

The following concepts and interpolation theorem for L_p will be very important in our treatment of singular integrals in the following section.

Definition 3.1. Let T be a mapping from L_p to L_q, where $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. Then T is of strong–type (p, q) if for $f \in L_p$

$$\|T\,f\|_q \leq A\|f\|_p, \tag{2}$$

where A is a constant not depending on f. Similarly, T is of weak–type (p, q) if

$$m\,(|T\,f| > \alpha) \leq \left(\frac{A\|f\|_p}{\alpha}\right)^q, \tag{3}$$

where $q < \infty$ and A does not depend on f or α, $\alpha > 0$. If $q = \infty$ strong–type and weak–type are the same.

It follows from straight application of Chebyshev inequality that strong–type (p, q) implies weak–type (p, q):

$$\alpha^q \, m\left(|T\, f| > \alpha\right) \leq \|T\, f\|_q^q \leq \left(A\, \|f\|_p\right)^q$$

One important idea in analysis is that of the splitting of a function that we will describe in the following proposition.

Proposition 3.1. *Let $1 \leq p_1 \leq p_2$. Then, $L_p \subset L_{p_1} + L_{p_2}$ for all $p_1 \leq p \leq p_2$.*

Proof. Fix $\gamma > 0$ and define

$$\begin{aligned} f_1(x) &= \begin{cases} f(x) & \text{for } x \in \{|f| > \gamma\} \\ 0 & \text{otherwise} \end{cases} \\ f_2(x) &= f(x) - f_1(x). \end{aligned}$$

Simple integration leads to

$$\begin{aligned} \int |f_1|^{p_1} = \int |f_1|^p |f_1|^{p_1 - p} &\leq \gamma^{p_1 - p} \int |f|^p \\ \int |f_2|^{p_2} = \int |f_2|^p |f_2|^{p_2 - p} &\leq \gamma^{p_2 - p} \int |f|^p. \end{aligned}$$

Hence, $f_1 \in L_{p_1}$ and $f_2 \in L_{p_2}$. □

Another application of the splitting technique is used in the following theorem due to Marcinkiewicz.

Theorem 3.2. (Marcinkiewicz) *Let $1 < r \leq \infty$. Suppose T is a sub-aditive map from $L_1 + L_r$ to the space of measurable functions on $\mathbb{R}^n$ which is simultaneously of weak–type $(1, 1)$ and weak–type (r, r). Then, T is of strong–type (p, p), for all $1 < p < r$. More explicitly, suppose that for all $f, g \in L_1 + L_r$*

(i) $|T\ (f + g)\ (x)| \leq |T\, f(x)| + |T\, g(x)|$
(ii) $m\left(|T\, f| > \alpha\right) \leq \frac{A_1}{\alpha}\, \|f\|_1, \qquad f \in L_1$
(iii) $m\left(|T\, f| > \alpha\right) \leq \left(\frac{A_r}{\alpha}\, \|f\|_r\right)^r$, *$f \in L_p$, and $r < \infty$.*

(If $r = \infty$ we assume that equation (2) holds). Then,

$$\|T\, f\|_p \leq A_p\, \|f\|_p, \qquad f \in L_p$$

for all $1 < p < r$, where A_p depends only on A_1, A_r, p and r.

Proof. We first consider the case $r < \infty$. Let $f \in L_p$ and define the function $\lambda(\alpha) = \{|T\, f| > \alpha\}$. As before, we can split $f = f_1 + f_2$, so that $f_1 \in L_1$ and $f_2 \in L_r$ by chopping $|f|$ at $\alpha > 0$. Condition (i) implies that

$$\{|T\, f| > \alpha\} \subset \{|T\, f_1| > \alpha/2\} \cup \{|T\, f_2| > \alpha/2\}.$$

Hence,

$$\lambda(\alpha) = m\,\{|T\, f| > \alpha\} \leq m\,\{|T\, f_1| > \alpha/2\} + m\,\{|T\, f_2| > \alpha/2\},$$

and by assumptions (ii) and (iii)

$$\lambda(\alpha) \leq \frac{2A_1}{\alpha} \int_{\mathbb{R}^n} |f_1(x)|\, dx + \frac{(2\, A_r)^r}{\alpha^r} \int_{\mathbb{R}^n} |f_2(x)|^r\, dx,$$

From the definition of f_1 and f_2, we conclude that

$$\lambda(\alpha) \leq \frac{2A_1}{\alpha} \int_{|f| > \alpha} |f(x)|\, dx + \frac{(2\, A_r)^r}{\alpha^r} \int_{|f| \leq \alpha} |f(x)|^r\, dx \tag{4}$$

Recall that

$$\int_{\mathbb{R}^n} |T\,f|^p\,dx = p\int_0^\infty \alpha^{p-1}\gamma(\alpha)\,d\alpha.$$

Multipying both sides of (4) by $p\,\alpha^{p-1}$ and integrating with respect to α gives

$$\begin{aligned}\int_0^\infty \alpha^{p-1}\alpha^{-1}\int_{|f|>\alpha}|f(x)|\,dx\,d\alpha &= \int_{\mathbb{R}^n}|f|\int_0^{|f|}\alpha^{p-2}\,d\alpha\,dx\\ &= \frac{1}{p-1}\int_{\mathbb{R}^n}|f||f|^{p-1}\,dx\end{aligned}$$

Similarly,

$$\begin{aligned}\int_0^\infty \alpha^{p-1}\alpha^{-r}\int_{|f|\le\alpha}|f(x)|^r\,dx\,d\alpha &= \int_{\mathbb{R}^n}|f|^r\int_{|f|}^{\infty}\alpha^{p-1-r}\,d\alpha\,dx\\ &= \frac{1}{r-p}\int_{\mathbb{R}^n}|f|^r|f|^{p-r}\,dx\end{aligned}$$

Consequently,

$$\|T\,f\|_p \le A_p\|f\|_p \quad \text{with} \quad (A_p)^p = \left(\frac{2\,A_1}{p-1}+\frac{(2\,A_r)^r}{r-p}\right)p.$$

Finally, we consider the case $r=\infty$. We decompose $f\in L_p$ as in proposition 4 taking $\gamma=\alpha/\left(2(A_\infty+1)\right)$. Then the following inequality holds (almost everywhere):

$$\begin{aligned}|T\,f| &\le |T\,f_1|+|T\,f_2|\\ &\le |T\,f_1|+\|T\,f_2\|_\infty\\ &\le |T\,f_1|+\frac{\alpha}{2}\end{aligned}$$

This means that $\{|T\,f|>\alpha\}\subset\{|T\,f_1|>\alpha/2\}$. Therefore

$$\begin{aligned}m\,\{|T\,f|>\alpha\} &\le m\,\{|T\,f_1|>\alpha/2\}\\ &\le \frac{2\,A_1}{\alpha}\int|f_1|\,dx\\ &= \frac{2\,A_1}{\alpha}\int_{2(A_\infty+1)|f|>\alpha}|f_1|\,dx\end{aligned}$$

Just as we did before, we multiply by $p\,\alpha^{p-1}$ both sides of the previous inequality, integrate with respect to α and apply Fubini's theorem to get:

$$\|T\,f\|_p\le A_p\|f\|_p \quad \text{with} \quad A_p^p=\frac{pA_1 2^p\,(A_\infty+1)^{p-1}}{p-1}$$

In this way we conclude the proof of the theorem. □

A result for linear operators in the same spirit will be given after in the following theorem

Theorem 3.3. **(M. Riesz)** *Let T be a linear operator which is both strong–types (p_0,q_0) and (p_1,q_1), so that*

$$\begin{aligned}\|T\,f\|_{q_0} &\le M_0\,\|f\|_{q_0} \qquad f\in L^{p_0}\\ \|T\,g\|_{q_1} &\le M_1\,\|g\|_{q_1} \qquad g\in L^{p_1}\end{aligned}$$

Then T is of strong–type (p_t, q_t) where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

and $0 \leq t \leq 1$. There exisst also a constant M_t, $0 \leq M_t \leq M_0^{1-t} M_1^t$, such that

$$\|T\,f\|_{q_t} \leq M_t\,\|f\|_{q_t} \qquad f \in L^{p_t}$$

To prove this theorem we will use the following Complex Analysis result [Rud87, p. 259].

Theorem 3.4. Phragmen–Lindelöf. *Suppose $\Omega = \{x + i\,y : a < x < b\}$, f is continuous on $\overline{\Omega}$, $f \in \mathbf{H}(\Omega)$, and suppose that $|f(z)| < B$ for all $z \in \Omega$ and some fixed $0 < B < \infty$. If*

$$M(x) = \sup\{|f(x + i\,y)| : -\infty < y < \infty\} \qquad (a \leq x \leq b)$$

then actually we have

$$M^{b-a}(x) \leq M^{b-x}(a) M^{x-a}(b) \qquad (a \leq x \leq b)$$

Proof. Without loss of generality assume $a = 0$, $b = 1$. For $\varepsilon > 0$ let

$$f_\varepsilon(z) = f(z) M(0)^{1-z} M(1)^{-z} \exp(\varepsilon z(z-1))$$

Then f_ε satisfies the conditions of the Theorem with $M(0) = M(1) = 1$. Observe that $|f_\varepsilon(z)| \to 0$ as $\text{Im}(z) \to \infty$; thus, $|f_\varepsilon(z)| \leq 1$ on a rectangle $[0,1] \times [-A, A] \subset \mathbb{C}$ for A large enough. It follows from the maximal principle theorem that $|f_\varepsilon(z)| \leq 1$ for all $z \in [0,1] \times \mathbb{R}$. Letting $\varepsilon \to 0$ we conclude that

$$|f(t + i\,y)| M(0)^{t-1} M(1)^{-t} = \lim_{\varepsilon \to 0} |f_\varepsilon(t + i\,y)| \leq 1, \quad 0 \leq t \leq 1$$

□

Proof of Theorem 4. Consider the set $\mathcal{D}$ of simple functions with finite measure steps. For any such $f = \sum_{j=1}^m a_j \mathbf{1}_{A_j} \in \mathcal{D}$ we have that $Tf \in L^{q_0} \cap L^{q_1}$, therefore $Tf \in L^{q_t}$ for any q_t between q_0 and q_1. From the Riesz representation Theorem we have that

$$\|Tf\|_{q_t} = \sup\{|g(Tf)| : g \in (L^{q_t})^* : \|g\|^* = 1\}$$

For $1 \leq t < \infty$, we know that $(L^{q_t})^* = L^{q_t'}$ and $\|g\|^* = \|g\|_{q_t'}$, where $1/q_t + 1/q_t' = 1$. Also, $g(Tf)$ is given by the expresion

$$g(Tf) = \int Tf(x)\, g(x)\, d\mu(x)$$

We will show that

$$|g(Tf)| \leq M_0^{1-t} M_1^t\, \|f\|_{p_t} \tag{5}$$

If $f = 0$ there is nothing to do. Otherwise, normalizing with $\|f\|_{p_t}$, we may assume that $\|f\|_{p_t} = 1$.

We will discuss the cases $p_t < \infty$ and $1 < q_t < \infty$ first, leaving the rest of the cases for the end. Since $\mathcal{D}$ is dense in any L^p, we only need to taste Tf with functions $g \in \mathcal{D}$. Let be $g = \sum_{k=1}^n b_k \mathbf{1}_{B_k} \in \mathcal{D}$ such that the conditions in the representation theorem are satisfied.

If $a_j = |a_j| e^{i\,\theta_j}$ and $b_k = |b_k| e^{i\,\theta_k}$, we define for each $z \in \mathbb{C}$ the following familly of functions in $\mathcal{D}$:

$$\begin{aligned} f_z &= \sum_{j=1}^{m} |a_j|^{\alpha(z)\, p_t} e^{i\,\theta_j} \mathbf{1}_{A_j} \\ g_z &= \sum_{k=1}^{n} |b_k|^{\beta(z)\, q_t'} e^{i\,\theta_k} \mathbf{1}_{B_k} \end{aligned}$$

where the functions α and β are given by the expressions

$$\alpha(z) = \frac{1-z}{p_0} + \frac{z}{p_1} \qquad \text{and} \qquad \beta(z) = \frac{1-z}{q_0} + \frac{z}{q_1}$$

In this way we can define an entire function F as

$$F(z) = \int (T f_z) g_z \, d\mu.$$

It is clear that the restriction of F to the strip

$$\{z = x + i\,y : 0 \leq x \leq 1\}$$

is bounded. A simple computation gives:

$$|f_{i\,y}|^{p_0} = \sum_{j=1}^{m} |a_j|^{p_t} \mathbf{1}_{A_j} = |f|^{p_t} = |f_{1+i\,y}|^{p_1} \tag{6}$$

$$|g_{i\,y}|^{q_0'} = \sum_{k=1}^{n} |b_k|^{q_t'} \mathbf{1}_{B_k} = |g|^{q_t'} = |g_{1+i\,y}|^{q_1'} \tag{7}$$

Application of Hölder's inequality and equations (6) and (7) lead to

$$\begin{aligned} |F(i\,y)| &\leq \|T f_{i\,y}\|_{q_0} \|g_{i\,y}\|_{q_0'} \\ &\leq M_0 \|f_{i\,y}\|_{p_0} \|g_{i\,y}\|_{q_0'} \\ &= M_0 \|f\|_{p_t}^{p_t/q_0} \|g\|_{q_t'}^{q_t'/q_0'} \\ &= M_0 \end{aligned}$$

and

$$\begin{aligned} |F(1+i\,y)| &\leq \|T f_{1+i\,y}\|_{q_1} \|g_{1+i\,y}\|_{q_1'} \\ &\leq M_1 \|f_{1+i\,y}\|_{p_1} \|g_{1+i\,y}\|_{q_1'} \\ &= M_1 \|f\|_{p_t}^{p_t/q_1} \|g\|_{q_t'}^{q_t'/q_1'} \\ &= M_1 \end{aligned}$$

The Phragmen–Lindelöf theorem implies that

$$F(t) \leq \sup\{|F(t+i\,y)| : t \in \mathbb{R}\} \leq M_0^{1-t} M_1^t$$

Note that $f_t = f$ and $g_t = g$ and define $M_t = \|Tf\|_{q_t}$. This shows that the statment of the theorem holds true for every $f \in \mathcal{D}$. To extend this result for any function in L^{p_t} we appoximate nonnegative functions in L^{p_t} by monotone sequences of functions in $\mathcal{D}$ and then apply monotone convergence.

The cases $(p_t = \infty, q_t = 1)$, $(p_t = \infty, q_t = \infty)$, $(p_t = 1, q_t = \infty)$, and $(p_t = 1, q_t = 1)$ are easy to deal with since they imply that $(p_t, q_t) = (p_r, q_r)$ for $r = 0$ or $r = 1$. For the case $(q_t = 1, p_t < \infty)$ we take $g_z = g$ for all z when we define $F(z)$; for the cases $(q_t > 1, p_t = 0)$ and $(q_t > 1, p_t = 1)$, we take $f_z = f$ in the definition of $F(z)$.
It remains to analize the case $(q_t = \infty, 0 < p_t < \infty)$. With no loss of generaliy, we can assume that $p_0 = q_0 = q_1 = \infty$ and $p_1 = 1$; thus, $1/p_t = t$. Let be $g \in (L^{q_t})^*$ with $\|g\|^* = 1$. Set f and f_z just as before and define F as $F(z) = g(Tf_z)$. Using the dual inequality $|g(Tf_z)| \leq \|Tf_z\|_\infty \|g\|^*$ and Hölders's inequality we obtain $\|F(1 + i\,y)| \leq M_1$. We get $|F(i\,y)| \leq M_\infty$ by noticing that $\|f_{i\,y}\|_\infty = \|\mathbf{1}_{\cup_j A_j}\|_\infty = 1$. □

The results on interpolation treated before work for any abstract measure spaces for which Fubini and Radon-Nikodym theorems hold. The same proofs developed before apply. Marcinkiewicz and Riesz theorems might be used to prove many interesting results. Here we present some examples:

Example 1. (Haussforff-Young) The function Transform $\mathcal{F}$ in $\mathbb{R}^n$ given by

$$\hat{f}(t) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot t} f(x)\, dx$$

is a linear operator which is of type $(1, \infty)$ and $(2, 2)$. In fact we have that

$$\begin{aligned} \|\mathcal{F}(f)\|_\infty &\leq \|f\|_1 \\ \|\mathcal{F}(f)\|_2 &= \|f\|_2 \end{aligned}$$

Thus by theorem 4, we can define the Fourier transform as an continuous operator $\mathcal{F} : L^{p_\theta} \longrightarrow L^{q_\theta}$, where $p_\theta = 2/(2-\theta)$ and $q_\theta = 2/\theta$ and $0 \leq \theta \leq 1$. so that $\|\mathcal{F}(f)\|_{q_\theta} \leq \|f\|_{p_\theta}$

Example 2.(Young's inequality) Let be $g \in L_1$ fixed. Then we define the convolution operator $T_g : f \longmapsto f * g$ then we have that

$$\begin{aligned} \|T_g\, f\|_1 &= \|f * g\|_1 \leq \|f\|_1 \|g\|_1 \qquad f \in L_1 \\ \|T_g\, f\|_\infty &= \|f * g\|_\infty \leq \|f\|_\infty \|g\|_1 \qquad f \in L_\infty \end{aligned}$$

By Riesz interpolation we get that T is of strong–type (p, p) for any p such that $1 \leq p \leq \infty$. Furthermore, the following inequality holds

$$\|T_g\, f\|_p = \|f * g\|_p \leq \|f\|_p \|g\|_1$$

By the same token, if we fix $f \in L_p$, we can define know the convolution operator $T_f g \longmapsto f * g$. We have that T_f is of type $(1, p)$ by the first part of this example. Let $g \in L^{p'}(\mathbb{R}^n)$ where $1/p + 1/p' = 1$. Then by the translation invariance of Lebesgue's measure and Hölder's inequality we have that

$$\|T_f\, g\|_\infty = \|f * g\|_\infty \leq \|f\|_p \|g\|_{p'}$$

thus, T_f is of type (p', ∞) as well. In this way we obtain the operator $T_f : L_p \longrightarrow L_r$ where $1/r = (1/p) + (1/q) - 1$ and

$$\|T_f\, g\|_r = \|f * g\|_r \leq \|f\|_p \|g\|_q$$

Example 3.(Hardy–Littlewood) The Hardy–Littlewood maximal function is of weak–type $(1,1)$ with

$$m\left(|M\,f| \geq \alpha\right) \leq \frac{3^n}{\alpha}\|f\|_1$$

and strong–type (∞, ∞), with

$$\|M\,f\|_\infty \leq \|f\|_\infty$$

Thus by the Marcinkiewicz theorem, it is of strong–type (p,p) for any p such that $1 < p < \infty$.

Example 4. Let μ be a complex Borel measure on $\mathbb{R}^n$. Let us define the linear operator $T : f \longmapsto f * \mu$. If $f \in L_1$, then by Fubini's theorem we have

$$\|Tf\|_1 \leq \|\mu\|\|f\|_1$$

where $\|\mu\|$ is the total variation of μ. If $f \in L^\infty(\mathbb{R}^n)$ then

$$\|Tf\|_\infty \leq \|\mu\|\|f\|_\infty$$

Both, Marcinkiewicz and Riesz theorems show that T is of strong–type (p,p) for any p such that $1 < p < \infty$. Furthermore, Riesz theorem gives

$$\|Tf\|_p \leq \|\mu\|\|f\|_p$$

In fact, it is not difficult to show for this example that $\|T\| = \|\mu\|$. We can elaborate more on this example. It is to be noted that the operator T just defined above also commutes with translations, $x \mapsto x + h$. A complete classification of all linear operators that commute with translations for the cases $p = 1$ and $p = 2$ ([DE00, Sec. 4] and [SW71, P. 19–40]) is given by the following results

Proposition 3.5. *Let T be a bounded linear transformation of L_1 into itself. Then T commutes with translations if and only if there exists a complex measure μ so that $T\,f = f * \mu$, for al $f \in L_1$. One has then $\|T\| = \|\mu\|$*

Proposition 3.6. *Let T be a bounded linear transformation of L_2 into itself. Then T commutes with translations if and only if there exists a function $b \in L^\infty(\mathbb{R}^n)$ so that $\widehat{Tf}(t) = b(t)\hat{f}(t)$, for all $f \in L_2$. One has then $\|T\| = \|b\|_\infty$*

For the special case in which T is both, L_1 and L_2 bounded, then we have that $b(t) = \hat{\mu}(t)$ where $\hat{\mu}(t)$, the Fourier transform of μ, is given by

$$\hat{\mu}(t) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot t}\, d\mu(x)$$

More applications, in particular of the Marcinkiewicz interpolation theorem, will arise in the treatment of singular integrals in the coming section, where we discuss the Calderón–Zigmund decomposition.

4. Singular Integrals

As we mention on example 4 at the end of the last section, the classification of continuous linear operators on L_1 and L_2 is well understood. For L_p in general, it seems that there is still work underway. However, for a class of convolution operators with singular kernel, with singularities at a point (the origin) and infinity, a lot of work has been done. The main idea is to get weak–type $(1,1)$ and $(2,2)$ estimates and then use interpolation theorems. Usually, the weak–type $(2,2)$ estimates are easy to obtain and harder work is require to get weak–type $(1,1)$ properties. We will follow a useful idea developed by Calderón and Zigmund. Basically, we divide the function in a good part, for which L_2 theory is applied, and a bad part where most of the difficulties are found. The simplest example of a singular integral is the Hilbert transform. This transform on a function f is defined as

$$H\,f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x-y)}{y}\,dy$$

where the non-absoluty convergent integral is defined by a suitable limiting process. Here we develop the general theory of Hilbert transform on L_p.

We start by stating the main theorem of this section and proceed to develop the ideas behind the proof.

Theorem 4.1. *Suppose the Kernel $K(x)$ satisfies the conditions*

$$|K(x)| \le B|x|^{-n} \qquad \forall\,|x| > 0 \tag{8}$$

$$\int\limits_{|x|\ge 2|y|} |K(x-y) - K(x)|\,dx \le B \forall\,|y| > 0 \tag{9}$$

and

$$\int\limits_{R_1<|x|<R_2} K(x)\,dx = 0 \qquad \forall\, 0 < R_1 < R_2 < \infty \tag{10}$$

For $f \in L_p$, $1 < p < \infty$ and any $\varepsilon > 0$, let

$$T_\varepsilon(f)(x) = \int\limits_{|y|\ge\varepsilon} f(x-y)K(y)\,dy. \tag{11}$$

Then

$$\|T_\varepsilon(f)\|_p \le A_p\|f\|_p \tag{12}$$

for some constant A_p independent of f and ε. Also, for each $f \in L_p$, $\lim_{\varepsilon\to 0} T_\varepsilon(f) = T(f)$ exists in L_p norm. The operator T so defined also satisfies inequality (12)

In order to prove theorem 4.1, we will need the following lemma that gives light to some properties of the kernel K.

Lemma 4.2. *Suppose K satisfies conditions (8), (9) and (10) stated above. Let K_ε be defined by*

$$K_\varepsilon(x) = \begin{cases} K(x) & \text{if } |x| \ge \varepsilon \\ 0 & \text{if } |x| < \varepsilon \end{cases}$$

Then obviously $K_\varepsilon \in L_2$*. For the fourier transforms, we have the following estimates*

$$\|\hat{K}_\varepsilon\|_\infty \leq CB \qquad \forall\, \varepsilon > 0$$

where C *depends only on the dimension* n

Proof. That $K_\varepsilon \in L_2$ follows from (8), and use of polar coordinates:

$$\int |K_\varepsilon(x)|^2\,dx = \int\limits_{|x|>\varepsilon} |K(x)|^2\,dx \leq B \int\limits_{|x|>\varepsilon} |x|^{-2n}\,dx = n^{-1}\varepsilon^{-n}\sigma_{n-1}$$

We claim that K_ε satisfies conditions (8), (9) and (10), as K does, exept that the bound B must be replaced by $B\,C$, where C depens only on dimension n: consider first the case $2|y| < \varepsilon$

$$\begin{aligned}\int\limits_{|x|\geq 2|y|} |K_\varepsilon(x-y) - K_\varepsilon(x)|\,dx = & \underbrace{\int\limits_{2|y|\leq|x|\leq\varepsilon} |K_\varepsilon(x-y) - K_\varepsilon(x)|\,dx}_{F(y)} \\ & + \underbrace{\int\limits_{\varepsilon<|x|} |K_\varepsilon(x-y) - K_\varepsilon(x)|\,dx}_{S(y)}\end{aligned}$$

For the first term we get

$$F(y) = \int\limits_{2|y|\leq|x+y|\leq\varepsilon} |K_\varepsilon(x)|\,dx \leq \int\limits_{\varepsilon\leq|x|\leq 3/2\,\varepsilon} |K(x)|\,dx \tag{13}$$

and for the second term we get

$$\begin{aligned} S(y) &= \int_{\left\{\begin{smallmatrix}\varepsilon \leq |x| \\ |x-y| \leq \varepsilon\end{smallmatrix}\right\}} |K(x)|\,dx + \int_{\left\{\begin{smallmatrix}\varepsilon \leq |x| \\ \varepsilon < |x-y|\end{smallmatrix}\right\}} |K(x-y) - K(x)|\,dx \\ &\leq \int\limits_{\varepsilon\leq|x|\leq 3/2\,\varepsilon} |K(x)|\,dx + \int\limits_{2|y|\leq|x|} |K(x-y) - K(x)|\,dx \end{aligned}$$

Using polar coordinates and conditions (8), (9) and (10) we get the following estimate

$$\begin{aligned}\int\limits_{|x|\geq 2|y|} |K_\varepsilon(x-y) - K_\varepsilon(x)| &\leq 2 \int\limits_{\varepsilon\leq|x|\leq 3/2\,\varepsilon} |K(x)|\,dx + B \\ &\leq B\,(2\sigma_{n-1}\log(3/2) + 1)\end{aligned}$$

The case $\varepsilon < 2|y|$ is treated in the same way:

$$\begin{aligned}\int\limits_{|x|\geq 2|y|} |K_\varepsilon(x-y) - K_\varepsilon(x)| &\leq \int_{\left\{\substack{|x|\geq 2|y| \\ |x-y|\leq\varepsilon}\right\}} |K(x)|\,dx \\ &\quad + \int_{\left\{\substack{|x|\geq 2|y| \\ |x-y|>\varepsilon}\right\}} |K(x-y)-K(x)|\,dx \\ &\leq \int\limits_{2|y|\leq|x|\leq 3|y|} |K(x)|\,dx + B \\ &< B\left(2\sigma_{n-1}\log(3/2)+1\right)\end{aligned}$$

Next,

$$\begin{aligned}\hat{K}_\varepsilon(y) &= \lim_{R\to\infty} \int\limits_{|x|\leq R} e^{-2\pi i x\cdot y} K_\varepsilon(x)\,dx \\ &= \int\limits_{\varepsilon\leq|x|\leq 1/|y|} e^{-2\pi i x\cdot y} K(x)\,dx + \lim_{R\to\infty} \int\limits_{1/|y|<|x|\leq R} e^{-2\pi i x\cdot y} K(x)\,dx \\ &= I_1 + I_2\end{aligned}$$

Condition (10) and polar coordinates lead us to

$$\begin{aligned}|I_1| &= \left| \int_{\varepsilon\leq|x|\leq 1/|y|} \left(e^{-2\pi i x\cdot y} - 1\right) K(x)\,dx \right| \\ &\leq 2\pi|y| \int\limits_{\varepsilon\leq|x|\leq 1/|y|} |x||K(x)|\,dx \\ &\leq 2\pi|y|B \int\limits_{\varepsilon\leq|x|\leq 1/|y|} |x|^{1-n}\,dx = 2\pi B\sigma_{n-1}\end{aligned}$$

To estimate I_2, we choose z such that $e^{-2\pi i z\cdot y} = -1$, say $z = \bar{y}/2|y|^2$, to get the identity

$$\int\limits_{1/|y|<|x|\leq R} e^{-2\pi i x\cdot y} K(x)\,dx = -\int\limits_{1/|y|<|x-z|\leq R} e^{-2\pi i x\cdot y} K(x-z)\,dx.$$

Thus, we get that $2\,I_2$ is given by

$$\lim_{R\to\infty} \left[\int\limits_{1/|y|<|x|\leq R} e^{-2\pi i x\cdot y} K(x)\,dx - \int\limits_{1/|y|<|x-z|\leq R} e^{-2\pi i x\cdot y} K(x-z)\,dx \right]$$

By adding and substracting $K(x-z)$ in the first integral of the righ t hand side we obtain the following decomposition for I_2:

$$
\begin{aligned}
2I_2 = \lim_{R\to\infty} \Bigg[& \int_{1/|y|\le|x|\le R} e^{-2\pi i x\cdot y} \left(K(x) - K(x-z)\right)\, dx \\
& + \int_{\left\{\begin{matrix} 1/|y| \le |x| \\ |x+z| < 1/|y| \end{matrix}\right\}} + \int_{\left\{\begin{matrix} |x| \le R \\ |x+z| > R \end{matrix}\right\}} e^{-2\pi i x\cdot y} K(x)\, dx \\
& - \int_{\left\{\begin{matrix} 1/|y| \le |x+z| \\ |x| < 1/|y| \end{matrix}\right\}} - \int_{\left\{\begin{matrix} |x+z| \le R \\ |x| > R \end{matrix}\right\}} e^{-2\pi i x\cdot y} K(x)\, dx \Bigg]
\end{aligned}
$$

The first term on the right hand side is bounded by our choice of z and condition (9). For the remaining terms we have the following estimates:

$$
\begin{aligned}
\left| \int_{\left\{\begin{matrix} 1/|y| \le |x+z| \\ |x| < 1/|y| \end{matrix}\right\}} e^{-2\pi i x\cdot y} K(x)\, dx \right| & \le \int\limits_{1/|y|\le|x|\le 3/2|y|} |K(x)|\, dx \\
& \le B\sigma_{n-1} \log(3/2) \\
\left| \int_{\left\{\begin{matrix} |x| < R \\ |x+z| > R \end{matrix}\right\}} e^{-2\pi i x\cdot y} K(x)\, dx \right| & \le \int\limits_{R-1/2|y|\le|x|\le R} |K(x)|\, dx \\
& \le B\sigma_{n-1} \log\left(\frac{R}{R - 1/2|y|}\right) \\
\left| \int_{\left\{\begin{matrix} 1/|y| \le |x+z| \\ |x| < 1/|y| \end{matrix}\right\}} e^{-2\pi i x\cdot y} K(x)\, dx \right| & \le \int\limits_{1/2|y|\le|x|\le 1/|y|} |K(x)|\, dx \\
& \le B\sigma_{n-1} \log(2) \\
\left| \int_{\left\{\begin{matrix} |x+z| < R \\ |x| > R \end{matrix}\right\}} e^{-2\pi i x\cdot y} K(x)\, dx \right| & \le \int\limits_{R\le|x|\le R+1/2|y|} |K(x)|\, dx \\
& \le B\sigma_{n-1} \log\left(\frac{R + 1/2|y|}{R}\right)
\end{aligned}
$$

The second and the fourth terms clearly converge to zero as R goes to infinity. Therefore the relationship $\|\hat{K}_\varepsilon\|_\infty \le B\sigma_{n-1}\,(2\pi + \log 3)$ for all $\varepsilon > 0$ holds. □

Proof of theorem 4.1: Plancherel's theorem and lemma 4.2 shows that the operator $T_\varepsilon(f)(x) = \int_{\mathbb{R}^n} f(x-y)K_\varepsilon(y)\,dy$ is of strong–type $(2,2)$. For any $f \in L_2$, the following is true:

$$m\left(|T_\varepsilon f| > \alpha\right) \le \frac{\|T_\varepsilon(f)\|_2^2}{\alpha^2} \le \frac{B^2 C_n^2}{\alpha^2}\|f\|_2^2 \tag{14}$$

Next, we show that T_ε is of weak–type $(1,1)$. To achieve this, we decompose f in two parts, $f = g + b$ where g stands for *good part* and b stands for *bad part.* Given $\alpha > 0$, consider the open set $\Omega = \{M\,f > \alpha\}$, where Mf is the Hardy-Littlewood maximal function. By the Maximal function theorem and theorem 2.1, $f(x) \le Mf(x) \le \alpha$ a.e on the closed set $F = \mathbb{R}^n \setminus \Omega$. Also, Ω can be decomposed by a sequence of cubes $\{Q_j\}$ with sides parallel to the axis and disjoint interiors whose diameters are comparable to their distances to F. Finally, the following relationships hold true:

$$m(Mf > \alpha) \le \frac{3^n}{\alpha}\|f\|_1 \quad \text{and} \quad \frac{1}{m(Q_j)}\int_{Q_j} |f(x)|\,dx \le K_n\alpha$$

where K_n is a constant depending only on dimension n. Let us define $g(x)$ a.e as follows:

$$g(x) = \begin{cases} f(x) & \text{for } x \in F \\ \frac{1}{m(Q_j)} \int\limits_{Q_j} f(t)\,dt & \text{for } x \in Q_j^o \end{cases}$$

The following are consequences of this definition

$$\begin{array}{ll} b(x) = 0 & \text{for } x \in F \\ \int\limits_{Q_j} b(x)\,dx = 0 & \text{for each cube } Q_j \end{array}$$

The linearity of each operator T_ε implies that

$$m\left(T_\varepsilon f > \alpha\right) \le m\left(T_\varepsilon g > \alpha/2\right) + m\left(T_\varepsilon b > \alpha/2\right) \tag{15}$$

The following estimates shows that $g \in L_2$

$$\begin{aligned} \|g\|_2^2 = \int_{\mathbb{R}^n} |g(x)|^2\,dx &= \int_F + \int_\Omega |g(x)|^2\,dx \\ &\le \alpha\int_F |f(x)|\,dx + K_n^2\alpha^2 m(\Omega) \\ &\le (1 + 3^n K^2(n))\alpha\|f\|_1 \end{aligned}$$

If we now apply inequality (14) to g, we obtain

$$m\left(|T_\varepsilon g| > \alpha/2\right) \le \frac{4C_n^2 B^2(1 + 3^n K_n^2)}{\alpha}\|f\|_1 \tag{16}$$

Next, we find an estimate for $T_\varepsilon b$. Let us define $b_j(x) = \mathbf{1}_{Q_j}(x)b(x)$. Then, $b(x) = \sum_j b_j(x)$ and $T_\varepsilon b(x) = \sum_j T_\varepsilon b_j(x)$.

For each cube Q_j, let be Q_j^* the cube with same center y_j and that has been expanded $2\sqrt{n}$ times. It can be easily seen that:

- : *(i)* $Q_j \subset Q_j^*$. If $\Omega^* = \cup Q_j^*$, then $\Omega \subset \Omega^*$ and $m(\Omega^*) \le 2^n n^{n/2} m(\Omega)$
- : *(ii)* If $F^* = \mathbb{R}^n \setminus \Omega^*$ then $F^* \subset F$
- : *(iii)* If $x \in \mathbb{R}^n \setminus Q_j^* = \left(Q_j^*\right)^c$, then $|x - y_j| \ge 2\sup\{|y - y_j| : y \in Q_j\}$
- : *(iv)* $\{|T_\varepsilon b| > \alpha/2\} \subseteq \{|\mathbf{1}_{F^*} T_\varepsilon b| > \alpha/2\} \cup \Omega^*$

The following is a simple consequence of the above:

$$m\left(|T_\varepsilon b| > \alpha/2\right) \le m\left(|\mathbf{1}_{F^*} T_\varepsilon b(x)| > \alpha/2\right) + \frac{6^n n^{n/2}}{\alpha}\|f\|_1$$

Since $\int_{Q_j} b_j(x)\,dx = \int_{Q_j} b(x)\,dx = 0$, we get

$$T_\varepsilon b_j(x) = \int\limits_{Q_j} \left[K_\varepsilon(x-y) - K_\varepsilon(x-y_j)\right] b_j(y)\,dy$$

Fubini's theorem leads to

$$\begin{aligned}
\int\limits_{F^*} |T_\varepsilon b(x)|\,dx &\le \sum_j \int\limits_{F^*} |T_\varepsilon b_j(x)|\,dx \\
&\le \sum_j \int\limits_{(Q_j^*)^c} \left| \int\limits_{Q_j} \left[K_\varepsilon(x-y) - K_\varepsilon(x-y_j)\right] b_j(y)\,dy \right| dx \\
&\le \sum_j \int\limits_{Q_j} |b_j(y)| \int\limits_{(Q_j^*)^c} |K_\varepsilon(x-y) - K_\varepsilon(x-y_j)|\,dx\,dy
\end{aligned}$$

For the last expression, let $y' = y - y_j$, then we have the following estimate:

$$\begin{aligned}
\int\limits_{(Q_j^*)^c} |K_\varepsilon(x-y) - K_\varepsilon(x-y_j)|dx &\le \int\limits_{|x-y_j|\ge 2|y'|} |K_\varepsilon(x-y) - K_\varepsilon(x-y_j)|dx \\
&= \int\limits_{|x|\ge 2|y'|} |K_\varepsilon(x-y') - K_\varepsilon(x)|dx \\
&\le B\,C_n
\end{aligned}$$

It follows inmidiately that

$$\int\limits_{F^*} |T_\varepsilon b(y)|\,dy \le B\,C_n \int\limits_{\Omega} |b(x)|\,dx \le 2B\,C_n\|f\|_1$$

The last inequality implies that

$$m\left(|\mathbf{1}_{F^*} T_\varepsilon b(x)| > \alpha/2\right) \le \frac{2}{\alpha}\|\mathbf{1}_{F^*} T_\varepsilon b\|_1 \le \frac{4B\,C_n}{\alpha}\|f\|_1,$$

therefore

$$m\left(|T_\varepsilon b| > \alpha/2\right) \le \frac{4B\,C_n + 6^n n^{n/2}}{\alpha}\|f\|_1 \tag{17}$$

Inequalities (15), (16) and (17) combined imply that

$$m\left(|T_\varepsilon f| > \alpha\right) \le \frac{D_n}{\alpha}\|f\|_1$$

uniformilly in ε ($D_n = D(n, B, C_n)$).

Marcinkiewickz interpolation theorem implies that T_ε is of (p,p) strong–type uniformilly on ε, with $1 < p \leq 2$. In other words, There exist a constant $A_p = A(n,p,B,C)$ such that for any $f \in L_p$,

$$||T_\varepsilon f\|_p \leq A_p\|f\|_p \quad (1 < p < 2)$$

The goal now is to extend this result to $p > 2$. The main idea is to follow a duality argument, noting that if $1/p + 1/q = 1$ then $1 < q < 2$. If a function ϕ is locally integrable and if $A = \sup\left\{\left|\int \phi\varphi\, dx\right| : \varphi \in \mathcal{C}_c(\mathbb{R}^n),\ \|\varphi\|_q = 1\right\}$ is finite, then $\phi \in L_p$ and $\|\phi\|_p = A$. Since $K_\varepsilon \in L_2$, it follows that for any $f \in L_1 \cap L_p$ with $2 < p < \infty$, and $\varphi \in \mathcal{C}_c$, $\|\varphi\|_q = 1$, the function $K_\varepsilon(x-y)f(y)\varphi(x) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Therefore, by Fubini's theorem we have that

$$\int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} K_\varepsilon(x-y)f(y)\, dy\right) \varphi(x)\, dx = \int\limits_{\mathbb{R}^n} f(y) \left(\int\limits_{\mathbb{R}^n} K_\varepsilon(x-y)\varphi(x)\, dx\right) dy$$

If we set $L_\varepsilon = K_\varepsilon(-x)$ then, the integral on parenthesis on the right hand side of the equation above is given by $L_\varepsilon * \varphi\,(y)$. Since L_ε satisfies the same conditions as K_ε, applying what we have shown for the cases we obtain that $\|L_\varepsilon * \varphi\|_q \leq A_q$. Therefore

$$\left|\int\limits_{\mathbb{R}^n} T_\varepsilon f(x)\, \varphi(x)\, dx\right| = \left|\int\limits_{\mathbb{R}^n} f(y)\, (L_\varepsilon * \varphi)(y)\, dy\right| \leq A_q\|f\|_p$$

Thus, $T_\varepsilon f \in L_p$ and $\|T_\varepsilon f\|_p \leq A_q\|f\|_p$. Since $L_1 \cap L_p$ is dense in L_p, the Carathe$\acute{o}$dory extension theorem does the rest.

Finally, we will prove that T_ε converges to some operator T as ε goes to 0. Let be $\varphi \in \mathcal{C}_c$ and assume $\varepsilon << 1$, then

$$T_\varepsilon\varphi(x) = T_1\varphi(x) + \int_{\varepsilon \leq |y| < 1} K(y)\left(\varphi(x-y) - \varphi(x)\right)\, dy$$

Let us denote the second term on the right hand side by $\Phi_\varepsilon(x)$. Since the convolution of an L^p function with an L^q function belongs to $\mathcal{C}_0$, we have that $T_1\varphi \in \mathcal{C}_o(\mathbb{R}^n)$ and $\Phi_\varepsilon \in L_p \cap \mathcal{C}_c(\mathbb{R}^n)$. If $\mathrm{supp}(\varphi) \subset B_a(0)$, then for any ε with $0 < \varepsilon < 1$, we have that $\mathrm{supp}(\Phi_\varepsilon) \subset B_{a+1}(0)$. Therefore, by use polar coordinates, we obtain the following inequalities

$$\begin{array}{rcl} |\Phi_{\varepsilon_1}(x) - \Phi_{\varepsilon_2}(x)| & \leq & B\, D_{n,\varphi}|\varepsilon_1 - \varepsilon_2|\, \sigma_{n-1} \\ \|\Phi_{\varepsilon_1} - \Phi_{\varepsilon_2}\ |_p & \leq & B\, D_{n,\varphi}|\varepsilon_1 - \varepsilon_2|\, \sigma_{n-1}\, \left(m(B_{a+1}(0))\right)^{1/p} \end{array}$$

This shows that Φ_ε, as a function of ε, is a Cauchy function on both the L_p and $\mathcal{C}_0(\mathbb{R}^n)$ norms. Hence, there exist a unique function $T\varphi \in \mathcal{C}_0(\mathbb{R}^n)$ to which $T_\varepsilon\varphi$ converges in L_p and $\mathcal{C}_0(\mathbb{R}^n)$, as ε goes to 0. Of course, $\|T\varphi\|_p \leq A_p\|\varphi\|_p$. The extension of T to all L_p follows inmidiately.

We can improve the result to get convergence a.e as well.

Corollary 4.3. *Let T_ε and T the linear operators defined in theorem 4.1. Then for any $f \in L_p$ we have that $\lim_{\varepsilon\to 0} T_\varepsilon f(x) = T\,f(x)$ a.e.*

Proof. Let be ε_k any sequence of positive numbers converging to 0 and let us define

$$\Lambda f(x) = \limsup_{k\to 0} |(T_{\varepsilon_k} - T)f(x)|$$

We will prove that $\Lambda f = 0$ a.e. This is certainly true for $f \in \mathcal{C}_c(\mathbb{R}^n)$ where $T_\varepsilon f$ converges uniformily to Tf in the $\mathcal{C}_0(\mathbb{R}^n)$ norm, as it was seen in the proof of theorem 4.1. For $f \in L_p$, choose a sequence of funtions $\varphi_j \in \mathcal{C}_c(\mathbb{R}^n)$ such that $\|f - \varphi_j\|_p \leq 1/j$ and let be $h_j = f - \varphi_j$. Let be $E(j,\lambda) = \{\Lambda h_j > \lambda\}$. It is clear that $\Lambda f \leq \Lambda h_j$, therefore

$$\{\Lambda f > \lambda\} \subset \bigcap_j E(j,\lambda) = E(\lambda)$$

Chebyshev's inequality and theorem 4.1 implies that

$$m\left(E(j,\lambda)\right) \leq \left(\frac{2\,A_p}{\lambda}\right)^p \|h_j\|_p^p \leq \left(\frac{2\,A_p}{\lambda\, j}\right)^p .$$

The rest follows inmidiately. □

We know that if T is an operator with kernel K, then it commutes with translations. Besides translations, there is another operation with the euclidean structure of $\mathbb{R}^n$, namely dilations. If $\varepsilon > 0$, let us denote by δ_ε, dilation by ε, the operator that maps the function $g(x)$ to $g(\varepsilon x)$. Thus, if T is an operator associated to the kernel $K(x)$ then $\delta_{\varepsilon^{-1}} T_{\delta_\varepsilon}$ is associated to the kernel $\varepsilon^{-n} K(\varepsilon^{-1} x)$. If T commutes with dilations, i.e, $T = \delta_{\varepsilon^{-1}} T_{\delta_\varepsilon}$ then $\varepsilon^{-n} K(\varepsilon^{-1} x) = K(x)$, in other words, K is homogeneous of degree $-n$. This means that for some homegeneous function Ω of degree 0

$$K(x) = \frac{\Omega(x)}{|x|^n} \tag{18}$$

This condition on Ω means that it is completely determined by its restriction to the unit sphere S^{n-1}. Our goal now, is to find the conditions we have to impose to Ω to get the analogous of theorem 4.1 for kernel of the form (18). Condition (8) implies that Ω should be bounded in S^{n-1}. Using polar coordinates we have that

$$\begin{aligned} \int\limits_{R_1<|x|<R_2} \frac{\Omega(x)}{|x|^n}\, dx &= \int\limits_{R_1}^{R_2} \int\limits_{S^{n-1}} \frac{\Omega(r\,u)}{r}\, d\sigma_{n-1}(u)\, dr \\ &= \log(R_1/R_2) \int\limits_{S^{n-1}} \Omega(u)\, d\sigma_{n-1}(u) \end{aligned}$$

thus, if condition (10) is to be held, we require that

$$\int_{S^{n-1}} \Omega(u)\, d\sigma_{n-1}(u) = 0 \tag{19}$$

Condition (9) is more subtle and sugests some sort of continuity on Ω. We introduce the following

Definition 4.1. A homogeneous function Ω of degree 0 is of *Dini Type* if

$$\int\limits_0^1 \frac{\omega(\delta)}{\delta}\, d\delta < \infty$$

where

$$\omega(\delta) = \sup\left\{ |\Omega(x) - \Omega(x')| : x, x' \in S^{n-1},\ |x - x'| \leq \delta \right\}$$

Observe that $\omega(\delta)$ is a non–decreasing function.
If K is as in (18) then

$$K(x-y)-K(x)=\left[\frac{\Omega(x-y)-\Omega(x)}{|x-y|^n}\right]+\Omega(x)\left[\frac{1}{|x-y|^n}-\frac{1}{|x|^n}\right]$$

For $|x|\geq 2|y|>0$ we have that the straight line segment from x to $x-y$ does not contained 0:

$$|x-ty|\geq |x|-t|y|\geq |x|-|y|\geq \frac{|x|}{2}\geq |y|$$

for all t such that $0\leq t\leq 1$. Let $\varphi(x)=|x|^{-n}$, then $\|\varphi'(x)\|=n|x|^{-(n+1)}$. By the mean value theorem and the observations above we get that

$$\int\limits_{|x|\geq 2|y|}|\varphi(x-y)-\varphi(x)|\,dx\leq n2^n\sigma_{n-1} \tag{20}$$

On the other hand,

$$\left|\frac{\Omega(x-y)-\Omega(x)}{|x-y|^n}\right|\leq \frac{2^n}{|x|^n}\left|\Omega\left(\frac{x-y}{|x-y|}\right)-\Omega\left(\frac{x}{|x|}\right)\right|$$

Using the triangle inequality several times we get

$$\left|\frac{x-y}{|x-y|}-\frac{x}{|x|}\right|\leq\left|\frac{x-y}{|x-y|}-\frac{x-y}{|x|}\right|+\frac{|y|}{|x|}\leq\frac{2|y|}{|x|}$$

Hence,

$$\begin{aligned}\int\limits_{|x|\geq 2|y|}\left|\frac{\Omega(x-y)-\Omega(x)}{|x-y|^n}\right|dx &\leq 2^n\int\limits_{|x|\geq 2|y|}\omega\left(\frac{2|y|}{|x|}\right)\frac{dx}{|x|^n}\\ &= 2^n\sigma_{n-1}\int\limits_{2|y|}^{\infty}\omega\left(\frac{2|y|}{r}\right)\frac{dr}{r}\\ &= 2^n\sigma_{n-1}\int\limits_{0}^{1}\frac{\omega(u)}{u}\,du\end{aligned}$$

Combining this estimate with (20), we obtained condition (9) of theorem 4.1:

$$\int\limits_{|x|\geq 2|y|}|K(x-y)-K(x)|\,dx\leq 2^n\sigma_{n-1}\left(\int\limits_0^1\frac{\omega(u)}{u}\,du+n\|\Omega\|_\infty\right)$$

This partially proves the following

Theorem 4.4. *Let Ω be a bounded, homogeneous of degree 0 function of Dini–type, satisfying condition (19) above. for $1<p<\infty$, and $f\in L_p$ let*

$$T_\varepsilon(f)(x)=\int\limits_{|y|\geq\varepsilon}\frac{\Omega(y)}{|y|^n}f(x-y)\,dy$$

: *(i)There exists $A_p>0$ (independent of f and ε such that*

$$\|T_\varepsilon(f)\|_p\leq A_p\|f\|_p$$

: *(ii)* $\lim\limits_{\varepsilon\to 0} T_\varepsilon(f) = T(f)$ *exists in* L^p *norm, and pointwise a.e.; moreover,*

$$\|T(f)\|_p \leq A_p \|f\|_p$$

: *(iii) If* $f \in L_2$ *, then there exists a bounded homogeneous function* m *of degree* 0 *such that*

$$(T(f))^\wedge(x) = m(x)\hat{f}(x)$$

Moreover, m *is given by the integral*

$$m(x) = \int\limits_{S^{n-1}} \left[\frac{\pi i}{2}\text{sign}\,(x\cdot y) + \log\left(\frac{1}{|x\cdot y|}\right)\right]\Omega(y)\,d\sigma_{n-1}(y) \tag{21}$$

for $|x| = 1$*.*

Proof. (i) and (ii) are consequences of theorem 4.1. While the existence of the bounded function m in the L_2 case, is guaranteed by proposition 3.6. That m is homogeneous of degree 0 follows from the fact that T commutes with dilations:

$$\begin{aligned}
\mathcal{F}(\delta_a T) &= a^{-n}\delta_{a^{-1}}(\mathcal{F}T) = a^{-n}(\delta_{a^{-1}}m)\cdot(\delta_{a^{-1}}\mathcal{F}) \\
\mathcal{F}(T\delta_a) &= m\cdot\mathcal{F}\delta_a = a^{-n}m\cdot(\delta_{a^{-1}}\mathcal{F})
\end{aligned}$$

Since $\delta_a T = T\delta_a$ then $m(ax) = m(x)$ for all $a > 0$. Formula (21) will be deduced as follows. Let

$$K_{\varepsilon,\eta}(x) = \begin{cases} \frac{\Omega(x)}{|x|^n} & \text{if } \varepsilon \leq |x| \leq \eta \\ 0 & \text{otherwise} \end{cases}$$

where $0 < \varepsilon < \eta < \infty$. Clearly $K_{\varepsilon,\eta} \in L_1 \cap L_2$. We will prove that

(a): $\sup|K^\wedge_{\varepsilon,\eta}(x)| \leq A$ for some $A > 0$ independent of ε and η.

(b): If $x \neq 0$, then

$$\lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}} K^\wedge_{\varepsilon,\eta}(x) = m(x)$$

The following lemma will come at handy.

Lemma 4.5. *Let* f *be a measurable function differentiable at 0. Assume that the integral* $\int\limits_1^\infty \frac{f(x)}{x}\,dx$ *converges. Then, for* $0 < a < b$*,*

$$\int\limits_0^\infty \frac{f(ax) - f(bx)}{x}\,dx = f(0)\log(b/a)$$

Proof.

$$\begin{aligned}
\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int\limits_\varepsilon^R \frac{f(ax)-f(bx)}{x}\,dx &= \int\limits_{a\varepsilon}^{aR}\frac{f(x)}{x}\,dx - \int\limits_{b\varepsilon}^{bR}\frac{f(x)}{x}\,dx \\
&= \int\limits_{a\varepsilon}^{b\varepsilon}\frac{f(x)}{x}\,dx - \int\limits_{aR}^{bR}\frac{f(x)}{x}\,dx
\end{aligned}$$

The second intentegral above converges to 0 when $R \to \infty$ since $\int_1^\infty \frac{f(x)}{x}\,dx$ converges. As for the fist integral we have that

$$\int_{a\varepsilon}^{b\varepsilon} \frac{f(x)}{x}\,dx = \int_{a\varepsilon}^{b\varepsilon} \frac{f(x)-f(0)}{x}\,dx + f(0)\log(b/a)$$

The differentiability of f at 0 implies that the integral on the right hand side of the last equation converges to 0 as $\varepsilon \to 0$. ☐

Ussing polar coordinates and letting $x = Ru$, with $u \in S^{n-1}$, we get

$$K^{\wedge}_{\varepsilon,\eta}(x) = \int_{S^{n-1}} \Omega(v) \left\{ \int_{\varepsilon}^{\eta} \frac{e^{-2\pi i R r u\cdot v}}{r}\,dr \right\} d\sigma_{n-1}(v).$$

We introduce the following auxiliary integral:

$$I_{\varepsilon,\eta}(x,v) = \int_{\varepsilon}^{\eta} \frac{\exp(-2\pi i R r u\cdot v) - \cos(2\pi R r)}{r}\,dr$$

Applying the theory of residues for instance, we can see that the imaginary part of the auxilary integral is uniformily bounded, and converges to

$$\operatorname{sign}(u\cdot v)\left(\int_0^\infty \frac{\sin t}{t}\,dt\right)$$

By lemma 4.5, with $f(x) = \cos x$, we have that the real part of the auxiliary integral is bounded by $C\log(1/|u\cdot v|) + C$ where C is a constant, and converges in fact to $\log(1/|u\cdot v|)$ as $\varepsilon \to 0$ and $\eta \to \infty$. Since $\int_{S^{n-1}} \Omega(v)\,d\sigma(v) = 0$, we can intruduce the factor $\cos(2\pi R r)$ in the integral defining $K^{\wedge}_{\varepsilon,\eta}$ to get

$$K^{\wedge}_{\varepsilon,\eta}(x) = \int_{S^{n-1}} I_{\varepsilon,\eta}(x,v)\Omega(v)\,d\sigma(v)$$

Because of the properties of $I_{\varepsilon,\eta}$ just mentioned

$$|K^{\wedge}_{\varepsilon,\eta}(x)| \le C'\|\Omega\|_\infty \int_{S^{n-1}} 1 + \log(1/|u\cdot v|)\,d\sigma(v)$$

Since $d\sigma_{n-1}$ is invariant under rotations, we have that

$$\begin{aligned}\int_{S^{n-1}} \log(1/|u\cdot v|)\,d\sigma(v) &= \int_{S^{n-1}} \log(1/|v_1|)\,d\sigma_{n-1}(v) \\ &= \sigma_{n-2}\int_0^\pi \log(|\sec\theta|)\sin^{n-2}(\theta)\,d\theta\end{aligned}$$

Since $\log(\sec\theta) \le e^{\frac{1}{\sqrt{\pi/2-\theta}}}$ for $0 \le \theta \le \pi/2$, dominated convergence implies (a)(uniform boundedness of $K_{\varepsilon,\eta}(x)$) and also we get

$$\lim_{\substack{\varepsilon \to 0 \\ \eta \to \infty}} K_{\varepsilon,\eta}(x) = m(x)$$

By Plancherel's theorem, if $f \in L^2$ then $K_{\varepsilon,\eta} * f$ converges in L^2 norm as $\eta \to \infty$ and $\varepsilon \to 0$, and the Fourier transform of this limit is $m(x)\hat{f}(x)$. On the other hand, if we fix ε and let $\eta \to \infty$, then clearly

$$\lim_{\eta\to\infty} \int_{\mathbb{R}^n} K_{\varepsilon,\eta}(y) f(x-y)\, dy = \int_{|y|>\varepsilon} K(y) f(x-y)\, dy$$

in L^2 norm, the limit being $T_\varepsilon(f)$.
Letting now $\varepsilon \to 0$, we obtain the conclusion (iii) and the teorem is proved. □

For the last theorem in this section, we will use the following result (See [DE00, Thm. 2.18], and [Ste70, pp. 62-63]).

Lemma 4.6. *Suppose that the leat decreasing radial majorant of* φ *is integrable; i.e, let* $\psi(x) = \text{ess.} \sup_{|y|\ge|x|} |\varphi(y)|$, *and* $\psi \in L_1$. *Then*

$$\sup_{\varepsilon>0} |(f * \varphi_\varepsilon)(x)| \le \|\psi\|_1 (M\, f)(x)$$

where $M\, f$ *is the Hardy's maximal function of* f.

Theorem 4.7. *Suppose that* Ω *satisfies the conditions of theorem 4.4. For* $f \in L_p$, $1 \le p < \infty$, *consider*

$$T_\varepsilon(f)(x) = \int_{|y|\ge\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)\, dy, \quad \varepsilon > 0$$

(The integral converges absolutely for every x.*)*

(i): Let $T^\star(f)(x) = \sup_{\varepsilon>0} |T_\varepsilon(f)(x)|$. *If* $f \in L_1$, *then the map* $f \mapsto T^\star f$ *is of weak type*$(1,1)$.

(ii): If $1 < p < \infty$, *then* $\|T^\star(f)\|_p \le A_p \|f\|_p$

Proof. We start by proving (ii) which is the easy part of the proof. For $f \in L_p$ the measurability of $T^\star f$ follows from continuity of $(T_\varepsilon f)(x)$ with respect to ε (when x is fixed):

$$|(T_{\varepsilon_1} f)(x) - (T_{\varepsilon_2} f)(x)| \le C_{n,p} \|f\|_p \left| \varepsilon_1^{n(1-q)} - \varepsilon_2^{n(q-1)} \right|$$

Let $\varphi \ge$ a smooth function whose support is contained in the unit ball and $\int \varphi\, dx = 1$. Let $\Phi = T\varphi - K_1$ where $T\varphi = \lim_{\varepsilon\to 0} T_\varepsilon \varphi$ in L^p. We show first that the least decreasing radial majorant of Φ is integrable.
For $|x| \le 1$ and $1 \le |x| \le 2$, we have that $\Phi(x) = T\varphi(x)$ and $\Phi(x) = T\varphi(x) - K_1(x)$

respectively. From the proof of theorem 4.1 and of corollary 4.3 $T\varphi$ is bounded. Hence, Φ is bounded on $B_2(0)$. For $|x| \geq 2$ note that

$$\Phi(x) = \int\limits_{\mathbb{R}^n} K(x-y)\varphi(y)\,dy - K(x) = \int\limits_{|y|\leq 1} \left[K(x-y) - K(x)\right]\varphi(x)\,dx$$

Using the estimates

$$\begin{array}{rcl} \left|\dfrac{\Omega(x-y)-\Omega(x)}{|x-y|^n}\right| & \leq & \dfrac{2^n}{|x|^n}\omega(2/|x|) \\ \left|\dfrac{1}{|x-y|^n} - \dfrac{1}{|x|^n}\right| & \leq & \dfrac{2^{n+1}n}{|x|^{n+1}} \end{array}$$

we obtain

$$|\Phi(x)| \leq C_n \left[\frac{1}{|x|^n}\omega\left(\frac{2}{|x|}\right) + \frac{1}{|x|^{n+1}}\right]$$

This shows that the least decreasing radial majorant of Φ is integrable. Since the operator T commutes with dilations, it follows that

$$\Phi_\varepsilon(x) = \varepsilon^{-n}\Phi(\varepsilon^{-1}x) = (T\varphi_\varepsilon)(x) - K_\varepsilon(x) \tag{22}$$

By the associativity of the convolution operator, for any $f \in L_p$

$$(T_\delta\varphi_\varepsilon) * f = (\varphi_\varepsilon * K_\delta) * f = (K_\delta * f) * \varphi_\varepsilon = (T_\delta f) * \varphi_\varepsilon$$

Passing to the limit as $\delta \to 0$ in L^p norm we get

$$(T\varphi_\varepsilon) * f = (T\,f) * \varphi_\varepsilon \tag{23}$$

Combining equations (22) and (23) we obtain

$$T_\varepsilon f = (T\,f) * \varphi_\varepsilon - f * \Phi_\varepsilon$$

If we denote by Ψ the least decreasing radial majorant of Φ, then by lemma 4.6

$$\sup_{\varepsilon>0} |(T_\varepsilon f)(x)| \leq (M\,Tf)(x) + \|\Psi\|_1(Mf)(x)$$

The L^p estimates for $f \mapsto f$ and of the Maximal function (example 3) proves (ii).

It remains to work on the more difficult part of the proof and which leads to (i). We follow the same steps as in the proof of weak–type (1,1) property of $T\,f$ in theorem 4.1. For given $\alpha > 0$ we split $f = g + b$ as before. We also consider for each cube Q_j the cube Q_j^* with same center y_j but whose sides have been expanded $2\sqrt{n}$ times. As we pointed out in the proof of theorem 4.1, if $x \in \left(Q_j^*\right)$ then

$$|x - y_j| \geq \operatorname{diam} Q_j \geq 2|y - y_j| \quad \text{for all } y \in Q_j$$

Let $y \in Q_j$, then

$$|x - y| \geq |x - y_j| - |y_j - y| \geq |y - y_j|$$

therefore, for any $z \in Q_j$

$$\begin{array}{rcl} |x - z| \leq |x - y| + |y - z| & \leq & |x - y| + \operatorname{diam} Q_j \\ & \leq & 2|x - y| + |y - y_j| \\ & \leq & 3|x - y| \end{array}$$

This shows that $Q_j \subset B_{3\varepsilon}(x) \setminus B_{\varepsilon/3}(x)$. Fix $x \in F^*$ and $\varepsilon > 0$. The cubes Q_j fall into three categories:

(a): cubes such that $Q_j \subset B_\varepsilon(x)$
(b): cubes such that $Q_j \subset (B_\varepsilon(x))^c$
(c): cubes such that $Q_j \cap \partial B_\varepsilon(x) \neq \emptyset$

We now examine

$$T_\varepsilon b(x) = \sum_j \int_{Q_j} K_\varepsilon(x-y) b(y)\, dy \tag{24}$$

Case (a) If Q_j belongs to the first class of cubes, then $K_\varepsilon(x-y) = 0$ so the integral over that cube in (24) is zero.

Case (b) If Q_j belongs to the second class of cubes, then $K_\varepsilon(x-y) = K(x-y)$ therefore, the integral over that cube equals

$$\int_{Q_j} K(x-y) b(y)\, dy = \int_{Q_j} [K(x-y) - K(x-y_j)] b(y)\, dy$$

This term is majorized by

$$\int_{Q_j} |K(x-y) - K(x-y_j)||b(y)|\, dy$$

Case (c) For the last class of cubes, we have that

$$\begin{aligned} \left| \int_{Q_j} K_\varepsilon(x-y) b(y)\, dy \right| &\leq \int_{Q_j} |K_\varepsilon(x-y)||b(y)|\, dy \\ &= \int_{B_{3\varepsilon}(x)\cap Q_j} |K_\varepsilon(x-y)||b(y)|\, dy \end{aligned}$$

Since

$$|K_\varepsilon(x-y)| \leq \left| \frac{\Omega(x-y)}{|x-y|^n} \right| \leq \frac{3^n B}{\varepsilon^n}$$

then

$$\int_{Q_j} |K_\varepsilon(x-y)||b(y)|\, dy \leq \frac{c_n}{m[B_{3\varepsilon}(x)]} \int_{B_{3\varepsilon}(x)\cap Q_j} |b(y)|\, dy$$

Adding over all cubes Q_j and taking the supremum over ε gives

$$|T^\star b(x)| \leq \sum_j \int_{Q_j} |K(x-y) - K(x-y_j)||b(y)|\, dy + c_n (M\, b)(x)$$

for all $x \in F^*$. Let us denote with Σ the first term on the right hand side. Then

$$m[|\mathbf{1}_{F^*} T^\star b| > \alpha/2] \leq m[\mathbf{1}_{F^*}\Sigma > \alpha/4] + m[c_n \mathbf{1}_{F^*}(M\, b) > \alpha/4]$$

As in the proof of theorem 4.1, the first summand on the right hand side is bounded by some expresion of the form $\frac{B_n}{\alpha}\|b\|_1$, whereas the second summand is bounded by a similar expresion because of the weak–type (1,1) property of the maximal function. We complete the proof by aplying (ii) to g, which is an L^2 function. ☐

5. Riesz Transforms and Spherical Harmonics

We discuss in this section a concrete example of the operators that we have analized before.

5.1. The Hilbert Transform. For the case $\mathbb{R}^1$ we take

$$Hf(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int\limits_{|y|>\varepsilon} \frac{f(x-y)}{y}\, dy$$

In other words $\Omega(x) = \frac{1}{\pi}\text{sign}\, x = \frac{1}{\pi}\frac{x}{|x|}$. For the L^2 case, according to theorem 4.4, $(Hf)^\wedge(x) = m(x)\hat{f}(x)$ with $m(x) = i\,\text{sign}\, x$. From this, it is clear that the Hilbert transform is a unitary operator and $H^2 = -I$.
The following proposition characterizes the Hilbert transform.

Proposition 5.1. *Suppose that T is a bounded operator on L_2 which satisfies the following properties:*

- *(a): T commutes with translations*
- *(b): T commutes with positive dilations*
- *(c): T anticommutes with the reflexion $f(x) \mapsto f(-x)$.*

Then T is a constant multiple of the Hilbert transform.

Proof. From (a), (b) and theorem 4.4, it follwos that $\mathcal{F}T = m{\cdot}\mathcal{F}$ where m is an homogeneous bounded function of degree 0. Let us denote by I the identity operator, then

$$\begin{aligned} \delta_a(m \cdot I) &= \delta_a(\mathcal{F}T\mathcal{F}^{-1}) = |a|^{-1}\mathcal{F}(\delta_{a^{-1}}(T\mathcal{F}^{-1})) \\ &= |a|^{-1}\text{sign}\,(a)\mathcal{F}T\delta_{a^{-1}}\mathcal{F}^{-1} = \text{sign}\,(a)\mathcal{F}T\mathcal{F}^{-1}\delta_a \\ &= \text{sign}(a)\, m \cdot I \end{aligned}$$

Hence$m(x) = m(1)\text{sign}\,(x)$. It follows that $\mathcal{F}T = m(1)\mathcal{F}H$. Taking inverse Fourier transform finishes the proof. ☐

The extension of the Hilber transform to $\mathbb{R}^n$ gives place to the *Riesz transformns.* Before we define this transforms, we should make a couple of remarks.

- *(i):* Any rotation ρ in $\mathbb{R}^n$ induces an action on functions, namely $\rho(f)(x) = f(\rho^{-1}x)$. A simple calculation shows that $\mathcal{F}\rho = \rho\mathcal{F}$.
- *(ii):* Let $m(x) = (m_1(x), \dots, m_n(x))$ be a function from $\mathbb{R}^n$ to itself. Suppose that for any rotation ρ in $\mathbb{R}^n$, m satifies the condition

$$m(\rho x) = \rho m(x) \tag{25}$$

In other words, if $\rho = (\rho_{jk})$ in the stardard basis, then

$$m_j(\rho x) = \sum_k \rho_{jk} m_k(x)$$

Lemma 5.2. *Suppose m is homogeneous of degree 0. If m satisfies (25) then, for some constant c*

$$m(x) = c\frac{x}{|x|}$$

That is

$$m_j(x) = c\frac{x_j}{|x|}$$

Proof. Suffices to consider $x \in S^{n-1}$. Let $c = m_1(e_1)$ and let ρ be any rotation that fixes e_1, i.e., $\rho e_1 = e_1$. Then for $j \geq 2$, $m_j(e_1) = \sum_{k=2}^{n} \rho_{jk} m_k(e_1)$. This implies that the vector $(m_2(x), \ldots, m_n(e_1))$ is fixed bay any rotation on $\mathbb{R}^{n-1}$. It follows that $m_j(e_1) = 0$ for $2 \leq j \leq n$.
For any $x \neq 0$, let ρ be any rotation such that $\rho e_1 = x$. Then $m(x) = m(\rho e_1) = \rho m(e_1) = c\rho e_1 = c\,x$. $\square$

Let c_n be a constant, which we will determine later, and let us define Ω_j as

$$\Omega_j(x) = c_n\frac{x_j}{|x|}$$

Clearly Ω_j is bounded and homogeneous of degree 0. If $x, x' \in S^{n-1}$ and $|x' - x| \leq \delta$ then

$$|\Omega_j(x') - \Omega_j(x)| = c_n|x'_j - x_j| \leq c_n\delta$$

Thus $\omega(\delta) \leq c_n\delta$ and the condition $\int_0^1 \frac{\omega(\delta)}{\delta}\,d\delta < \infty$ is satisfied. Finally,

$$\begin{aligned}\int\limits_{S^{n-1}} \Omega_j(u)\,d\sigma_{n-1}(u) &= c_n\sigma_{n-2}\int\limits_0^{\pi} \cos\theta\sin^{n-2}\theta\,d\theta \\ &= -c_n\sigma_{n-2}\int\limits_{-\pi/2}^{\pi/2} \sin\theta\cos^{n-2}\theta\,d\theta = 0\end{aligned}$$

We are now in the position to define the Riesz transforms R_j for each $j = 1, \ldots, n$.

Definition 5.1. For any $f \in L_p$ with $1 \leq p < \infty$,

$$R_j(f)(x) = \lim_{\varepsilon \to 0} c_n \int\limits_{|y| \geq \varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y)\,dy$$

where $c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}$

Thus R_j is defined by the kernel $K_j(x) = \frac{\Omega_j(x)}{|x|^n}$ with $\Omega_j(x) = c_n\frac{x_j}{|x|}$.

Let $M(t) = \frac{\pi i}{2}\operatorname{sign} t + \log(1/|t|)$. When we consider the transforms R_j with $j = 1, \ldots, n$ to be defined on L_2, we know from theorem 4.4 of the existance of a multipliers $m_j(x)$ given by

$$m_j(x) = \int\limits_{S^{n-1}} M(x \cdot y)\Omega_j(y)\,d\sigma_{n-1}(y) \quad \|x| = 1$$

Since $d\sigma_{n-1}$ is rotation invariant, then for any rotation ρ we have that

$$\begin{aligned} m_j(\rho x) &= \int\limits_{S^{n-1}} M(\rho x \cdot y)\Omega_j(y)\, d\sigma_{n-1}(y) \\ &= \int\limits_{S^{n-1}} M(x \cdot \rho^{-1} y)\Omega_j(\rho\rho^{-1}y)\, d\sigma_{n-1}(y) \\ &= \int\limits_{S^{n-1}} M(x \cdot u)\Omega_j(\rho u)\, d\sigma_{n-1}(u) \\ &= \int\limits_{S^{n-1}} M(x \cdot u) \sum_{k=1}^{n} \rho_{jk}\Omega_k(u)\, d\sigma_{n-1}(u) \\ &= \sum_{k=1}^{n} \rho_{jk} m_k(x) \end{aligned}$$

Thus by lemma 5.2 we have that if $m(x) = (m_i(x), \dots, m_n(x))$ then

$$m(x) = m(e_1)\frac{x}{|x|} = i\frac{x}{|x|}$$

as a simple computatin shows.

It is very easy to verufy that for any rotation ρ

$$\rho^{-1}R_j(\rho f) = \sum_{k=1}^{n} \rho_{jk} R_k(f).$$

The following theorem gives a characterization of the Riezs transforms.

Theorem 5.3. *Let $T = (T_1, \dots, T_n)$ be an n-tuple of bounded operators in L_2. Suppose that*

(i): Each T_j commutes with translations
(ii): Each T_j commutes with dilations
(iii): For each rotation $\rho = (\rho_{jk})$ of $\mathbb{R}^n$,

$$\rho^{-1}T_j(\rho f) = \sum_{k=1}^{n} \rho_{jk} T_k(f)$$

Then there is a constant c such that $T_j = cR_j$ $j = 1, \dots, n$.

Proof. (i) and (ii) imply that there are bounded homegeneous functions m_j of degree 0 such that $\mathcal{F}T_j = m_j \cdot \mathcal{F}$. (iii) implies that

$$\mathcal{F}(\rho^{-1}T_j(\rho f)) = \sum_{k=1}^{n} \rho_{jk}(\mathcal{F}T_k(f))$$

The left hand side of the equation above equals $(\rho^{-1}m_j) \cdot \hat{f}$, and the right hand side equals $(\sum_{k=1}^{n} \rho_{jk} m_k) \cdot \hat{f}$. Therefore $m(\rho x) = \rho m(x)$ which, by lemma 5.2, implies that

$$m_j(x) = c\frac{x_j}{|x|}$$

for some constant c. □

5.2. Two applications of Riesz Transformations.

Proposition 5.4. *Suppose $f \in L_2$ has all derivatives of order 2 in $L^p \cap L^2$ with $1 < p < \infty$. Let*

$$\Delta f = \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_j^2}$$

the Laplace operator. Then we have the a priori bound

$$\left\| \frac{\partial^2 f}{\partial x_j \partial x_k} \right\|_p \leq A_p \|\Delta f\|_p$$

Proof. Using Fourier transform we obtain

$$\begin{aligned} \left(\frac{\partial^2 f}{\partial x_j \partial x_k} \right)^{\wedge} (x) &= -4\pi^2 x_j x_k \hat{f}(x) \\ &= -\left(\frac{i x_j}{|x|} \right) \left(\frac{i x_k}{|x|} \right) (-4\pi^2 |x|^2) \hat{f}(x) \\ &= -(R_j R_k \Delta f)^{\wedge}(x) \end{aligned}$$

This gives that

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = -R_j R_k \Delta f$$

Theorem 4.4 applies. ☐

Proposition 5.5. *Suppose f has all its derivatives of orden 1 in $L^p \cap L^2$ of $\mathbb{R}^2$ with $1 < p < \infty$. Then we have the a priori bound*

$$\left\| \frac{\partial f}{\partial x_1} \right\|_p + \left\| \frac{\partial f}{\partial x_2} \right\|_p \leq A_p \left\| \frac{\partial f}{\partial x_1} + i \frac{\partial f}{\partial x_2} \right\|_p$$

Proof. We only need to check that

$$\frac{\partial f}{\partial x_j} = -R_j (R_1 - i R_2) \left(\frac{\partial f}{\partial x_1} + i \frac{\partial f}{\partial x_2} \right)$$

☐

5.3. Spherical Harmonics.

Definition 5.2. The space of solid spherical harmonics of degree k, denoted as $\mathcal{H}_k$ is the linear space of homogeneous harmonic polynomials of degree k

Example 5
$P(x,y) = x^3 y - x y^3$ is a solid spherical harmonic of degree 4.
$Q(x,y) = x^2 - y^2$ is a solid spherical harmonic of degree 2.

We restric $\mathcal{H}_k$ to the unit sphere S^{n-1} and define the inner product

$$(P|Q) = \int_{S^{n-1}} P(x) \overline{Q(x)} \, d\sigma_{n-1}(x)$$

The following lemma (Euler) will be needed:

Lemma 5.6. *If F is homogeneous of degree k and differentiable at the point x, then*

$$\nabla F(x) \cdot x = kF(x)$$

Proof. Let $\varphi(t) = F(t\,x) = t^k F(x)$ and apply the chain rule. ☐

Using the lemma above and Green's theorem we can show that the finite dimensional spaces $\{\mathcal{H}_k\}_{k\in\mathbb{N}}$ are mutually orthogonal:

$$\begin{aligned}(k-j)\int\limits_{S^{n-1}} P\overline{Q}d\,d\sigma &= \int\limits_{S^{n-1}} kP\overline{Q}\,d\sigma - \int\limits_{S^{n-1}} P\overline{jQ}\,d\sigma \\ &= \int\limits_{S^{n-1}} \nabla P\cdot x\,\overline{Q}\,d\sigma - \int\limits_{S^{n-1}} P\overline{\nabla Q}\cdot x\,d\sigma \\ &= \int\limits_{S^{n-1}} (\overline{Q}\nabla P - P\overline{\nabla Q})\cdot x\,d\sigma \\ &= \int\limits_{B(0;1)} \nabla\cdot(\overline{Q}\nabla P - P\overline{\nabla Q})\,dx = 0\end{aligned}$$

The following theorem gives a decomposition of the space of homogeneous polinomials of degree k in terms of solid spherical harmonics. ☐

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