

DEPENDENCE ON INITIAL CONDITIONS AND PARAMETERS

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1. INTRODUCTION

It is very typical that the systems of differential equations used to model physical phenomena depend on certain parameters inherent to the model. These parameters may appear explicitly in the equation or may be given as initial conditions.

In many instances, the functions that are used to model a physical situation may be approximations or simplifications to more complicated equations.

The purpose of this set of notes is to discuss the local sensitivity of a system of ode's to slight variation on the parameters.

2. EXISTENCE AND UNIQUENESS IN THE PRESENCE OF PARAMETERS

In many applications, one consider differential equations of the form

$$\dot{x} = f(t, x, \lambda) \tag{1}$$

where $f : \Omega \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^d \longrightarrow \mathbb{R}^n$ satisfies certain smoothness conditions, and where λ represent constants or implicit or explicit parameters that are inherent to the model. Examples of explicit parameters are: mass of particles, gravitation in mechanical systems; rate of birth, rates

of death in population models. An example of implicit parameters is initial conditions.

We first generalize the Theorem of existence and uniqueness of solutions to (1) under some smooth conditions on f .

Theorem 2.1. *Consider the differential equation (1), where f is an \mathbb{R}^n -valued continuous function defined in some domain $\Omega \subset \mathbb{R}^{1+n+d}$. Suppose that for each point $(t, x, \lambda) \in \Omega$, there are numbers $a, b, p > 0$ such that*

$$\|f(t, x, \lambda) - f(t, y, \lambda)\| \leq L\|x - y\|$$

in the box $[t - a, t + a] \times \bar{B}_b(x) \times \bar{B}_p(\lambda) \subset \Omega$.

Then, for each $(t_0, x_0, \lambda_0) \in \Omega$, there are $r, p > 0$ and a continuous function $x : [t_0 - r, t_0 + r] \times \bar{B}_p(\lambda_0) \rightarrow \mathbb{R}^n$ differentiable on t such that

- (i) x solves (1) with $x(t_0, \lambda) = x_0$ for all $\lambda \in \bar{B}_p(\lambda_0)$
- (ii) $(t, x(t, \lambda)) \in \Omega$ for all $t \in (t_0 - r, t_0 + r) \times \bar{B}_p(\lambda_0)$.

Proof. Except for a slight modification, we repeat verbatim the the proof presented in class to the uniqueness and existence theorem of ode's.

By hypothesis, there are constants $a, b, p > 0$ such that in the \mathbb{R}^{1+n+d} -box

$$\Gamma = [t_0 - a, t_0 + a] \times \bar{B}_b(x_0) \times \bar{B}_p(\lambda_0) \subset \Omega$$

f is continuous and satisfies a Lipschitz continuous on x . Thus, for some constants M and L we have

$$\sup_{(t,x,\lambda) \in \Gamma} \|f(t, x, \lambda)\| \leq M,$$

and for all $(t, \lambda) \in [t_0 - a, t_0 + a] \times \bar{B}_p(\lambda_0)$ and $x, y \in \bar{B}_b(x_0)$

$$\|f(t, x, \lambda) - f(t, y, \lambda)\| \leq L\|x - y\|$$

Choose $r > 0$ such that $r \leq \min(a, b/M)$ and $r < 1/L$. Notice that the uniform ball

$$\mathcal{D}_b(x_0) = \{x \in C([t_0 - r, t_0 + r] \times \bar{B}_p(\lambda_0), \mathbb{R}^n) : \|x - x_0\|_u \leq b\}$$

is a closed subset of the space of \mathbb{R}^n -valued continuous functions on the box $[t_0 - r, t_0 + r] \times \bar{B}_p(\lambda_0)$, hence it is complete.

Define the function $\Psi : \mathcal{D}_b(x_0) \rightarrow C([t_0 - r, t_0 + r] \times \bar{B}_p(\lambda_0), \mathbb{R}^n)$ by

$$\Psi(y)(t, \lambda) := x_0 + \int_{t_0}^t f(s, y(s, \lambda), \lambda) ds$$

Notice that $\Psi(t)(t_0, \lambda) = x_0$ for all $y = y(t, \lambda) \in \mathcal{D}_b(x_0)$.

Our choice of r implies

(a) Ψ maps the ball $\mathcal{D}_b(x_0)$ to itself, to wit

$$\begin{aligned} \|\Psi(y)(t, \lambda) - x_0\| &= \left\| \int_{t_0}^t f(s, y(s, \lambda), \lambda) ds \right\| \\ &\leq \left| \int_{t_0}^t \|f(s, y(s, \lambda), \lambda)\| ds \right| \\ &\leq M|t - t_0| \leq M r \end{aligned}$$

(b) Ψ is a contraction on $\mathcal{D}_b(x_0)$. To wit, let $x, y \in \mathcal{D}_b(x_0)$, then

$$\begin{aligned} \|(\Psi(x) - \Psi(y))(t, \lambda)\| &= \left\| \int_{t_0}^t f(s, x(s, \lambda), \lambda) - f(s, y(s, \lambda), \lambda) ds \right\| \\ &\leq \left| \int_{t_0}^t \|f(s, x(s, \lambda), \lambda) - f(s, y(s, \lambda), \lambda)\| ds \right| \\ &\leq L \left| \int_{t_0}^t \|x(s, \lambda) - y(s, \lambda)\| ds \right| \\ &\leq Lr \|x - y\|_u \end{aligned}$$

An application of the Banach fixed point Theorem shows that there is a unique $x \in \mathcal{D}_b(x_0)$ such that $\Psi(x)(t, \lambda) = x(t, \lambda)$, and the Theorem follows. \square

Although Theorem 2.1 presents minimal conditions for existence and uniqueness of solutions as well as continuous dependence on parameters, it is natural to expect that (1) depends smoothly on parameters (f is differentiable in λ). In this situation, we can study the sensitivity of the solution $x(t, \lambda)$ to slight variations on the value of λ .

3. DEPENDENCE ON PARAMETERS

To motivate the study of the effect of changes in the values of parameters in a differential equation, we present the following example

Example 3.1. Consider the one-dimensional ode

$$\dot{x}(t, \lambda) = \lambda x^2(t, \lambda) + t \tag{2}$$

Notice that if $\lambda_0 = 0$, (2) becomes

$$\dot{x} = t$$

for which the initial condition $x(0) = 0$, has solution $x(t; 0) = \frac{t^2}{2}$. For small values of λ , we may want to see how different $x(t; \lambda)$ is from the

real solution $x(t; \lambda)$. If we use a Taylor expansion on λ for $x(t, \lambda)$ we get the following first order approximation:

$$x(t, \lambda) \approx x(t, 0) + \frac{\partial x}{\partial \lambda}(t, 0)\lambda \quad (3)$$

The chain rule applied on (2) implies that

$$\frac{\partial \dot{x}}{\partial \lambda}(t, 0) = \frac{t^4}{4}$$

This is the variational equation to (2). This equation has solution $\frac{\partial x}{\partial \lambda}(t, 0) = \frac{t^5}{20}$. By (3) we get the approximation

$$x(t, \lambda) \approx \frac{t^2}{2} + \lambda \frac{t^5}{20}$$

3.1. Comparison results. In order to investigate the effect of changes of solutions due to variations on the value of initial conditions or parameters, we introduce the following result

Proposition 3.1. *(Gronwall) Let α , φ , and ψ be non negative continuous functions defined on the interval $[a, b]$. Moreover, assume that α is differentiable and that $\dot{\alpha}(t) \geq 0$. If for all $t \in [a, b]$*

$$\varphi(t) \leq \alpha(t) + \int_a^t \psi(s)\varphi(s) ds \quad (4)$$

then,

$$\varphi(t) \leq \alpha(t) \exp \left[\int_a^t \psi(s) ds \right] \quad (5)$$

for all $t \in [a, b]$.

Proof. The following proof is taken from [Chi99, p. 128], although the proof is very similar to the one in [Ver00, p. 5] when $\alpha \equiv \text{constant}$.

We only consider the case $\alpha(t) > 0$, as the general case follows from this case by considering $\alpha(t) + \varepsilon$ for $\varepsilon > 0$ small.

Notice that (4) implies that

$$\frac{\varphi(t)}{\alpha(t) + \int_a^t \psi(s)\varphi(s) ds} \leq 1$$

Multiplying both sides of this equation by ψ and then adding and subtracting $\dot{\alpha}$ in the numerator of the resulting inequality produces the following estimate

$$\frac{\psi(t)\varphi(t) + \dot{\alpha}(t)}{\alpha(t) + \int_a^t \psi(s)\varphi(s) ds} \leq \frac{\dot{\alpha}(t)}{\alpha(t)} + \psi(t) \quad (6)$$

Integrating (6) over $[a, t]$ and exponentiation gives

$$\alpha(t) + \int_a^t \psi(s)\varphi(s) ds \leq \alpha(t) \exp \left[\int_a^t \psi(s) ds \right]$$

Inequality (5) follows from the hypothesis (4). \square

A particular useful form of Gronwall's inequality is

Corollary 3.2. *Let $\delta_j \geq 0$ ($j = 1, 2, 3$) be constants and $\varphi(t)$ a non negative function over $[a, b]$. Suppose that*

$$\varphi(t) \leq \delta_2(t - a) + \delta_1 \int_a^t \varphi(s) ds + \delta_3 \quad (7)$$

for all $t \in [a, b]$. Then,

$$\varphi(t) \leq \left(\frac{\delta_2}{\delta_1} + \delta_3 \right) e^{\delta_1(t-a)} - \frac{\delta_2}{\delta_1} \quad (8)$$

Proof. Inequality (7) can be rewritten as

$$\varphi(t) + \frac{\delta_2}{\delta_1} \leq \left(\frac{\delta_2}{\delta_1} + \delta_3 \right) + \delta_1 \int_a^t \left(\psi(s) + \frac{\delta_2}{\delta_1} \right) ds$$

Then, by Proposition 3.1 with $\varphi + \frac{\delta_2}{\delta_1}$, $\delta_3 + \frac{\delta_2}{\delta_1}$ and δ_1 instead of φ , α and ψ respectively, (8) follows. \square

The first application of this specialized Gronwall inequality is the comparison between solutions of differential equations that are very *closed* to each other [Sán79, p. 140].

Theorem 3.3. *Consider the pair of differential equations*

$$\begin{aligned} \dot{x} &= f(t, x) & \dot{y} &= g(t, y) \\ x(t_0) &= x_0 & y(t_0) &= y_0 \end{aligned}$$

where f and g are assume to be continuous in some domain $\Omega \subset \mathbb{R}^{1+n}$, and for all $(t, x) \in \Omega$

$$\|f(t, x) - g(t, x)\| \leq \varepsilon$$

and

$$\|x_0 - y_0\| \leq \delta$$

for some $\varepsilon, \delta > 0$ small. Furthermore, assume that f satisfies the Lipschitz condition

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

Then, there exists an interval (m, M) such that

$$\|x(t) - y(t)\| \leq \left(\delta + \frac{\varepsilon}{L}\right) e^{L|t-t_0|} - \frac{\varepsilon}{L} \quad (9)$$

for all $t \in (m, M)$.

Proof. We present a sketch of the proof. The existence of such interval (m, M) follows from the existence and uniqueness Theorem.

Inequality (9) follows from

$$\begin{aligned} \|x(t) - y(t)\| &\leq \|x_0 - y_1\| + \left\| \int_{t_0}^t f(s, x(s)) - g(s, y(s)) ds \right\| \\ &\leq \delta + \left| \int_{t_0}^t \|f(s, y(s)) - g(s, y(s))\| ds \right| \\ &\quad + \left| \int_{t_0}^t \|f(s, x(s)) - f(s, y(s))\| ds \right| \\ &\leq \delta + \varepsilon|t - t_0| + L \left| \int_{t_0}^t \|x(s) - y(s)\| ds \right|, \end{aligned}$$

and the *specific* Gronwall's Lemma 3.2. \square

3.2. Variational equations. Recall the differential equation (1)

$$\dot{x} = f(t, x, \lambda)$$

We will use $x(t) = x(t; \mathbf{y}, \lambda)$ to denote the solution to (1) that satisfies the initial condition $x(t_0; \mathbf{y}, \lambda) = \mathbf{y}$.

If f satisfies the conditions of Theorem 2.1, then not only does the solution $x(t; \mathbf{y}, \lambda)$ exist locally, but also it is continuous in (t, λ) . In fact, Theorem 3.3 clearly implies that $x(t; \mathbf{y}, \lambda)$ is continuous in (t, \mathbf{y}, λ) . If we further assume that $f = f(t, x, \lambda)$ is continuously differentiable w.r.t. (x, λ) then, it can be shown through Theorem 3.3 that $x(t; \mathbf{y}, \lambda)$ is differentiable in (t, \mathbf{y}, λ) . See [Per91, §2.3] for a proof for dependence on initial conditions.

A simple application of the chain rule imply that the rates of change, w.r.t. initial conditions and parameters, of solutions to (10) satisfy themselves certain differential equations. Indeed, given any function $G(t, y, \lambda)$, we will use the notation

$$\begin{aligned} \dot{G}(t, y, \lambda) &= \frac{dG}{dt}(t, y, \lambda) \\ G_y(t, y, \lambda) &= \frac{\partial G}{\partial y}(t, y, \lambda) \\ G_\lambda(t, y, \lambda) &= \frac{\partial G}{\partial \lambda}(t, y, \lambda) \end{aligned}$$

We have the following result:

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^{1+n+d}$ and assume that f is continuously differentiable in Ω . If $(t_0, x_0, \lambda_0) \in \Omega$, then there exist $r, b, \rho > 0$ such that for each $(\mathbf{y}, \lambda) \in B_b(x_0) \times B_\rho(\lambda_0)$, the initial value problem*

$$\dot{x} = f(t, x, \lambda) \quad (10)$$

$$x(t_0) = \mathbf{y} \quad (11)$$

has a unique solution $\mathbf{x}(t; \mathbf{y}, \lambda)$, which is continuously differentiable in the box $(t_0 - r, t_0 + r) \times B_b(x_0) \times B_\rho(\lambda_0)$.

Furthermore, if

$$\Phi(t, \mathbf{y}, \lambda) = x_{\mathbf{y}}(t; \mathbf{y}, \lambda)$$

$$\Psi(t, \mathbf{y}, \lambda) = x_{\lambda}(t; \mathbf{y}, \lambda),$$

the $\dot{\Phi}$ and $\dot{\Psi}$ satisfy the initial value differential equations

$$\dot{\Phi}(t, \mathbf{y}, \lambda) = f_{\mathbf{y}}(t, x(t; \mathbf{y}, \lambda), \lambda)\Phi(t, \mathbf{y}, \lambda) \quad (12)$$

$$\Phi(t_0, \mathbf{y}, \lambda) = I$$

where I is the $n \times n$ -identity matrix, and

$$\dot{\Psi}(t, \mathbf{y}, \lambda) = f_{\lambda}(t, x(t; \mathbf{y}, \lambda), \lambda)\Psi(t, \mathbf{y}, \lambda) + f_{\lambda}(t, x(t; \mathbf{y}, \lambda), \lambda) \quad (13)$$

$$\Psi(t_0, \mathbf{y}, \lambda) = \mathbf{0}$$

where $\mathbf{0}$ is the $n \times d$ -zero matrix.

Sketch of the proof. Equations (12) and (13) follow from a formal application of the chain rule.

To prove the differentiability of $x(t; \mathbf{y}, \lambda)$ w.r.t \mathbf{y} and λ , we first notice, by Theorem 2.1 and Theorem 3.3, that $f(t, x(t, \mathbf{y}, \lambda), \lambda)$ is continuous in some box contained in Ω . We then use the formal derivatives in (12) and (13) to compare $x(t; \mathbf{y}, \lambda)$ and $x(t; \mathbf{y} + \mathbf{h}, \lambda + k)$. Theorem 3.3 is then used to show that the residual in the formal expression for the derivative is indeed of order $o(\|h, k\|)$ uniformly along some small interval around t_0 . \square

Equations (12) and (13) are known as the the variational equations with respect to initial conditions and parameters respectively.

3.3. Homework.

Problem 3.1. Solve the initial value problem

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}) \\ \mathbf{x}(0) &= (y_1, y_2, y_3)^\top \end{aligned}$$

for $f(x_1, x_2, x_3) = (-x_1, -x_2 + x_1^2, x_3 + x_1^2)^\top$. Denote the solution by $\mathbf{x}(t; \mathbf{y})$ and compute

$$\Phi(t, \mathbf{y}) = \mathbf{x}_y(t; \mathbf{y})$$

By direct computation, show that

$$\begin{aligned}\dot{\Phi} &= f_y(t, \mathbf{y})\Phi \\ \Phi(0, \mathbf{y}) &= I\end{aligned}$$

REFERENCES

- [Chi99] Carmen Chicone. *Ordinary differential equations with applications*, volume 34 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1999.
- [Per91] Lawrence Perko. *Differential equations and dynamical systems*, volume 7 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1991.
- [Sán79] David A. Sánchez. *Ordinary differential equations and stability theory*. Dover Publications Inc., New York, 1979. An introduction, Reprint of the 1968 original.
- [Ver00] Ferdinand Verhulst. *Nonlinear differential equations and dynamical systems*. Universitext. Springer-Verlag, Berlin, second edition, 2000. Revised and Extended.

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