POTENTIAL THEORY AND MARKOV CHAINS

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Contents

1. Markov transition functions and Martingale problem

- Let $(S, \mathscr{B})$ be a separable complete metric space with its Borel σ–algebra $\mathscr{B}$.
- Let P be a Markov transition probability function (t.p.f.) on $(S, \mathscr{B})$, and let $L = P - I$ its ***generator***.
- We will use $\{X_n\}$ to denote the Markov chain on S associated to P, so that

$$\mathbb{E}_x[f(X_n)] := \int f(y)P^n(x, dy) = P^n f(x)$$

- The ***shifting*** operator θ on path-space $S^{\mathbb{Z}_+}$ is defined by $X_n(\theta(\omega)) = X_{n+1}(\omega)$ for all $n \in \mathbb{Z}_+$. The ***natural*** history of X is the ***filtration*** $\{\mathscr{F}_n\}$, $\mathscr{F}_n = \sigma(X_m : m \leq n)$.
- Recall that $\{X_n, \mathscr{F}_n : n \in \mathbb{Z}_+\}$ defines a Markov process on ***path space*** $(S^{\mathbb{Z}_+}, \mathscr{B}^{\otimes \mathbb{Z}_+})$ if and only if, for any bounded measurable function f on $(S, \mathscr{B})$,

$$\mathbb{E}_x[f(X_{t+s})|\mathscr{F}_t] = \mathbb{E}_{X_t}[f(X_s)] = P^s(f(X_t))$$

- Given a bounded measurable function h, we define a transition kernel I_h in S by

$$I_h f(x) := h(x)f(x)$$

Theorem 1.1. *Let $\{X_n, \mathscr{F}_n : n \in \mathbb{Z}_+\}$ be an adapted process. X is a Markov process with transition probability function P if and only if for all bounded measurable function f on $(S, \mathscr{B})$*

$$M_t := f(X_t) - f(X_0) - \sum_{s=0}^{t-1} Lf(X_s), \qquad M_0 = 0$$

is a martingale w.r.t. $\{\mathscr{F}_t\}$.

Proof. Necessity: Suppose $\{X_n, \mathscr{F}_n\}$ is a Markov process with t.p.f P. Then, by the Markov property

$$\begin{aligned}
\mathbb{E}[M_{t+1}|\mathscr{F}_t] &= \mathbb{E}\left[f(X_{t+1}) - f(X_0) - \sum_{s=0}^{t} Lf(X_s)\big|\mathscr{F}_t\right] \\
&= Pf(X_t) - f(X_0) - \sum_{s=0}^{t-1} Lf(X_s) - Lf(X_t) \\
&= M_t + Pf(X_t) - Lf(X_t) - f(X_t) = M_t
\end{aligned}$$

Sufficiency: Suppose $\{M_n, \mathscr{F}_n\}$ is a martingale. Then

$$\mathbb{E}[f(X_{t+r})|\mathscr{F}_r] = M_t + f(X_0) + \mathbb{E}\Big[\sum_{s=0}^{t-1} Lf(X_s)\big|\mathscr{F}_r\Big] \tag{1}$$

$$+ \mathbb{E}\Big[\sum_{s=0}^{r-1} Lf(X_s \circ \theta_t)\big|\mathscr{F}_t\Big] \tag{2}$$

$$= f(X_t) + \mathbb{E}\Big[\sum_{s=0}^{r-1} Lf(X_s \circ \theta_t)\big|\mathscr{F}_t\Big] \tag{3}$$

Define $L = P - I$ and notice that for $r = 1$, (1) takes the form

$$\begin{aligned}
\mathbb{E}\big[f(X_{t+1})|\mathscr{F}_t\big] &= f(X_t) + \mathbb{E}\big[Lf(X_t)|\mathscr{F}_t\big] \\
&= f(X_t) + Lf(X_t) = Pf(X_t).
\end{aligned}$$

By induction, assume that $\mathbb{E}[g(X_{t+s})|\mathscr{F}_t] = P^s g(X_t)$ for all bounded g and $1 \leq s < r$. Then,

$$\begin{aligned}
\mathbb{E}[f(X_{t+r})|\mathscr{F}_t] &= f(X_t) + \mathbb{E}\Big[\sum_{s=0}^{r-1} Lf(X_{s+t})\big|\mathscr{F}_t\Big] = f(X_t) + \sum_{s=0}^{r-1} \big(P^s L\, f\big)(X_t) \\
&= f(X_t) + \sum_{s=0}^{r-1} \big(P^s(P-I)\, f\big)(X_t) \\
&= f(X_t) + \sum_{s=0}^{r-1} \big(P^{s+1} - P^s\big) f(X_t) = P^r(X_t)
\end{aligned}$$

Therefore, X is a Markov process with t.p.f. P. $\square$

2. Operators associated to Markov operators

- For any $A \in \mathscr{B}$, define the stopping times
$$\sigma_A = \inf\{n \geq 0 : X_n \in A\}; \tag{4}$$
called ***first hitting time*** of A, and
$$\tau_A := \inf\{n > 0 : X_n \in A\}; \tag{5}$$
called ***first return time*** to A. Denote by $L(x, A) = \mathbb{E}_x[\tau_A < \infty]$.
- Observe that $\sigma_A \circ \theta = \tau_A - 1$
- The point process
$$\eta(A) = \sum_{n=1}^{\infty} \delta_{X_n}(A), \qquad A \in \mathscr{B}$$
counts the number of visits of the chain X to a given set A.
- The expected number of returns of the chain X starting at x, $A \mapsto U(x, A) = \mathbb{E}_x[\eta(A)]$, defines random kernel from S to itself.

Lemma 2.1. *$L(x, A) > 0$ if and only if $U(x, A) > 0$*

Proof. From the definition of U, it follows that $U(x, A) > 0$ if and only if $\mathbb{P}_x[\eta(A) > 0] > 0$. Since
$$\mathbb{1}_{\{\tau_A < \infty\}} = \mathbb{1}_{\{\eta(A) > 0\}} = \mathbb{1}_{\bigcup_{n \geq 1}\{X_n \in A\}},$$
the conclusion follows immediately. □

- Suppose T is a random variable in $\mathbb{Z}_+$ and that T is independent from the chain X. If $a(n) = \mathbb{P}[T = n]$, $n \in \mathbb{Z}_+$, then
$$K_a(x, A) := \mathbb{P}_x[X_T \in A] = \sum_{n=0}^{\infty} a(n) P^n(x, A)$$
is a Markov kernel on $(S, \mathscr{B})$.

Theorem 2.2. *Let $\varepsilon > 0$ and $A \in \mathscr{B}$. Assume that $\sum_n a(n) = 1$ and $a(n) > 0$ for all $n \in \mathbb{Z}_+$. Then, $L(x, A) > 0$ for all x iff $K_a(x, A) > 0$ for all x.*

Proof. Necessity: Clearly $U(x, A) > 0$ implies that $K_a(x, A)$ since $a(n) > 0$ for all n.

Sufficiency: Observe that $K_a(y, A) > 0$ for all $y \in S \setminus A$ implies that $L(y, A) > 0$ for all $y \in S \setminus A$. By the Markov property,
$$\begin{aligned} L(x, A) &= \mathbb{P}_x[X_1 \in A] + \mathbb{P}_x[X_1 \in A^c, \tau_A < \infty] \\ &= P(x, A) + \int_{S \setminus A} L(y, A) P(x, dy). \end{aligned}$$
Therefore, $L(x, A) > 0$ for all x. □

- Give a stopping time T, we define a transition kernel P_T in S by
$$P_T f(x) := \mathbb{E}_x[f(X_T)\mathbb{1}_{\{T<\infty\}}]$$
- Let $D \in \mathscr{B}$ and μ be a function on S such $\mu(x)+1 \neq 0$ for any $x \in S$.

2.1. **Balayage operator.**

- The balayage (scanning) operator P_{μ,D^c} is defined by
$$P_{\mu,D^c} = \sum_{n\geq 0} \big(I_{\phi \mathbb{1}_D} P\big)^n I_{D^c} \tag{6}$$
where $\phi = (1+\mu)^{-1}$.
- Given a bounded measurable function g in S, it is easy to check that
$$P_{\mu,D^c} g(x) = \mathbb{E}_x\big[g(X_{\sigma_{D^c}}) \prod_{k=1}^{\sigma_{D^c}} (1+\mu(X_k))^{-1}\,\mathbb{1}_{\{\sigma_{D^c}<\infty\}}\big]$$
- It is easy to check that
$$(I_{\phi\mathbb{1}_D} P) P_{\mu,D^c} = P_{\mu,D^c} - I_{D^c} \tag{7}$$
- From (7) it follows that $h(x) = P_{\mu,D^c} g(x)$ solves the ***Poisson*** boundary problem
$$\begin{array}{rcll} \mu(x)h(x) - Lh(x) & = & 0 & x \in D \\ h(x) & = & g(x) & x \in D^c \end{array}$$

Example 2.1. Suppose $S = \Gamma \cup \Gamma^c$, $\Gamma^c = A \cup D$, where $A \cap D = \emptyset$. The solution to the Poisson problem
$$\begin{array}{rcll} (\mu - L)h(x) & = & 0 & x \in \Gamma \\ h(x) & = & \mathbb{1}_A(x) & x \in \Gamma^c \end{array}$$
denoted by $h^\mu_{A,D}$ is called the μ–***capacitor*** of A with respect to D.
Hence, $h^\mu_{A,D}(x) = P_{\mu,\Gamma^c}\mathbb{1}_A(x)$ and clearly
$$\begin{array}{rcl} h^\mu_{A,D}(x) & = & \mathbb{E}_x\Big[(1+\mu)^{-\sigma_A}; \sigma_A < \sigma_D\Big] \\ & = & \mathbb{1}_A(x) + \mathbb{1}_\Gamma(x)\mathbb{P}_x\big[(1+\mu)^{-\tau_A}; \tau_A < \tau_D\big] \end{array}$$

2.2. **Killing operator.**

- The killing operator G_{μ,D^c} is defined as
$$G_{\mu,D^c} = \sum_{n\geq 0}\big(I_{\phi\mathbb{1}_D}P\big)^n I_{\phi\mathbb{1}_D} = \sum_{n\geq 0} I_{\phi\mathbb{1}_D}\big(P I_{\phi\mathbb{1}_D}\big)^n$$
where $\phi = (1+\mu)^{-1}$.
- Given a bounded measurable function g in S, it is easy to check that
$$G_{\mu,D^c} g(x) = \mathbb{E}_x\Big[\sum_{k=0}^{\sigma_{D^c}-1} g(X_k) \prod_{j=0}^{k} (1+\mu(X_j))^{-1}\Big]$$

- It is easy to check that
$$(I_{\phi \mathbb{1}_D} P) G_{\mu, D^c} = G_{\mu, D^c} - I_{\phi \mathbb{1}_D} \tag{8}$$
- From (8) it follows that $f(x) = G_{\mu, D^c} g(x)$ solves the ***Dirichlet*** problem
$$\begin{aligned} \mu(x) f(x) - L f(x) &= g(x) \quad x \in D \\ f(x) &= 0 \qquad x \in D^c \end{aligned}$$

Example 2.2. Let $w_A(x) := \mathbb{E}_x[\sigma_A]$. Observing that
$$w_A(x) = \mathbb{E}_x\Big[\sum_{j=0}^{\sigma_A - 1} \mathbb{1}(X_j) \Big] = G_A \mathbb{1}(x),$$
we have that w_A solves the Dirichlet problem
$$\begin{aligned} -Lw(x) &= \mathbb{1}(x) \quad x \in A^c \\ w(x) &= 0 \qquad x \in A \end{aligned}$$

2.3. **Taboo operator.**

- The taboo operator U_{μ, D^c} is defined as
$$U_{\mu, D^c} = \sum_{n \geq 1} \big(P I_{\phi \mathbb{1}_D} \big)^{n-1} P = \sum_{n \geq 1} P \big(I_{\phi \mathbb{1}_D} P \big)^{n-1}$$
where $\phi = (1+\mu)^{-1}$.
- For any bounded measurable function g on S it is easy to check that
$$U_{\mu, D^c} g(x) = \mathbb{E}_x \Big[\sum_{n=1}^{\tau_{D^c}} g(X_n) \prod_{j=1}^{n-1} (1 + \mu(X_j))^{-1} \Big]$$
- It is clear that
$$U_{\mu, D^c} I_{D^c} g(x) = \mathbb{E}_x \Big[g(X_{\tau_{D^c}}) \prod_{j=1}^{\tau_{D^c} - 1} (1 + \mu(X_j))^{-1} \mathbb{1}_{\{\tau_{D^c} < \infty\}} \Big]$$
- The balayage and taboo operators are related by the equation
$$P_{\mu, D^c} = I_{D^c} + I_{\phi \mathbb{1}_D} U_{\mu, D^c} I_{D^c}$$

2.4. **Potential operator.**

- The μ–potential operator with respect to P is defined by
$$G_\mu = \sum_{n \geq 0} \big(I_\phi P \big)^n I_\phi$$
where $\phi = (1+\mu)^{-1}$.
- Given a bounded measurable function g on S, we have that
$$G_\mu g(x) = \mathbb{E}_x \Big[\sum_{k \geq 0} g(X_k) \prod_{j=0}^{k} \big(1 + \mu(X_j) \big)^{-1} \Big]$$

- It is easy to check that

$$I_\phi P G_\mu = G_\mu - I_\phi$$

- It follows that $f(x) = G_\mu g(x)$ satisfies

$$\big(\mu(x) - L\big)h(x) = g(x)$$

3. Maximum Principle

Combining the killing and balayage operators we obtain the following result

Theorem 3.1. *Let $c : D \longrightarrow [0,\infty)$ and $\phi : D^c \longrightarrow [0,\infty)$ measurable. If $\mathbb{P}_x[\sigma_{D^c} < \infty] = 1$ for all $x \in S$, then*

$$h(x) = G_{D^c}c(x) + P_{D^c}\phi(x) = \mathbb{E}\Big[\sum_{k=0}^{\sigma_{D^c}-1} c(X_k) \ + \ \phi(X_{\sigma_{D^c}})\mathbb{1}_{\{\sigma_{D^c}<\infty\}}\Big] \tag{9}$$

is a unique solution to the problem

$$\begin{array}{rcll} -Lh(x) & = & c(x) & x \in D \\ h(x) & = & \phi(x) & x \in D^c \end{array}$$

The following results are known as the ***maximal principles*** for Markov chains

Theorem 3.2. *Let c and ϕ and h as in Theorem 3.1. If $u : S \longrightarrow [0,\infty)$ satisfies*

$$\begin{array}{rcll} -Lu(x) & \geq & c(x) & x \in D \\ u(x) & \geq & \phi(x) & x \in D^c \end{array} \tag{10}$$

then, $u(x) \geq h(x)$ for all $x \in S$.

Proof. For each $n \in \mathbb{Z}_+$ define the nonnegative sequence

$$h_n(x) = \mathbb{E}_x\Big[\sum_{k=0}^{n-1} c(X_k)\mathbb{1}_{\{k<\sigma_{D^c}\}} + \phi(X_{\sigma_{D^c}})\mathbb{1}_{\{\sigma_{D^c}<n\}}\Big]$$

Observe that $h_0 \equiv 0$ and that

$$h_n(x) = \Big(\sum_{k=0}^{n-1} I_D\big(PI_D\big)^k c\Big)(x) + \Big(\sum_{k=0}^{n-1} \big(I_D P\big)^k I_{D^c}\phi\Big)(x)$$

It follows immediately that $h_{n+1}(x) = Ph_n(x) + c(x)$ for all $x \in D$ and $h_{n+1}(x) = \phi(x)$ for all $x \in D^c$; moreover, by monotone convergence, $h_n \nearrow h$. If u satisfies (10), then $h \geq h_0 \equiv 0$; and by induction, $u \geq u_n$ implies $u \geq Pu + c \geq Ph_n + c = h_{n+1}$ on D and $u \geq \phi = h_{n+1}$ on D^c. Therefore, $u \geq \lim_n h_n = h$. $\square$

Theorem 3.3. *Suppose $\{X_n, \mathscr{F}_n\}$ is a Markov process with t.p.f P. Let D be a bounded open domain and assume that $\mathbb{P}[\sigma_{D^c} < \infty] \equiv 1$. If u is a bounded nonnegative functions such that $Pu(x) \geq u(x)$ on D, then*

$$\sup_{x\in D} u(x) \leq \sup_{x\in D^c} u(x)$$

Proof. We first show that $\{u(X_n^{\sigma_D}), \mathscr{F}_n\}$ is a submartingale. Indeed,

$$M_t = u(X_t) - u(X_0) - \sum_{s=0}^{t-1} Lu(X_s), \qquad M_0 = 0$$

is a martingale. Hence,

$$\begin{aligned} u(X_n^{\sigma_{D^c}}) &= M_n^{\sigma_{D^c}} + u(X_0) + \sum_{s=0}^{\sigma_{D^c}\wedge n-1} Lu(X_s) \\ &\geq M_n^{\sigma_{D^c}} + u(X_0) + \sum_{s=0}^{\sigma_{D^c}\wedge(n-1)-1} Lu(X_s) \\ &= M_n^{\sigma_{D^c}} - M_{n-1}^{\sigma_{D^c}} + u(X_{n-1}^{\sigma_{D^c}}) \end{aligned}$$

Since $(M_n^{\sigma_{D^c}}, \mathscr{F}_n)$ is a martingale, then

$$\mathbb{E}_x\big[u(X_n^{\sigma_{D^c}})|\mathscr{F}_{n-1}\big] \geq u(X_{n-1}^{\sigma_{D^c}}).$$

Consequently, $\{u(X_n^{\sigma_{D^c}}), \mathscr{F}_n\}$ is a submartingale. Hence,

$$\mathbb{E}_x\big[u(X_{\sigma_{D^c}\wedge n})\big] \geq \mathbb{E}_x[u(X_{\sigma_{D^c}\wedge 0})] = u(x)$$

By Doob's optional time theorem and dominated convergence we conclude that

$$\mathbb{E}_x[u(X_{\sigma_{D^c}})] \geq u(x), \qquad x \in S.$$

Therefore, for any $x \in D$, $\sup_{y\in D^c} u(y) \geq \mathbb{E}_x[u(X_{\sigma_{D^c}}] \geq u(x)$. □

4. Dual process

- For any σ–finite measure on $(S, \mathscr{B})$ we use the notation

$$mf := \int f\,dm, \qquad \langle f, g\rangle_m := \int f\overline{g}\,dm, \qquad mP\,f := \int Pf\,dm.$$

- m is an ***invariant*** probability measure for P if

$$mP\,f := m\,f \qquad f \in L_1(m).$$

- It is easy to see that $\|Pf\|_2 \leq \|f\|_2$ for any $f \in \mathcal{L}_2(m)$, for

$$\begin{aligned} \int \Big| \int f(y)P(x,dy)\Big|^2 m(dx) &\leq \int\int |f(y)|^2 P(x,dy)\,m(dx) \\ &= \int P|f|^2(x)\,m(dx) = \|f\|^2_{L_2(m)}. \end{aligned}$$

- The ***adjoint operator*** P^* is the the unique operator on $\mathcal{L}_2(m)$ such that
$$\langle Pf, g\rangle_m = \langle f, P^* g\rangle_m, \qquad f,\, g \in \mathcal{L}_2(m)$$
- It follows immediately that $P^*\mathbb{1} = 1$.

Lemma 4.1. *Let* $\mathscr{G}_n = \sigma(X_k : k \geq n)$. *Then*
$$\mathbb{E}_m[f(X_n)|\mathscr{G}_{n+1}] = P^* f(X_{n+1})$$

Proof. Let f and g be bounded measurable functions on $(S, \mathscr{B})$ and on $(S^{\mathbb{Z}_+}, \mathscr{B}^{\otimes\mathbb{Z}_+})$ respectively. Let $\psi(x) = \mathbb{E}_x[g(X_0, X_1, \dots)] = \mathbb{E}_x[g(X)]$, then by the Markov property
$$\begin{aligned}
\mathbb{E}_m[f(X_n)\overline{g(X\circ\theta_{n+1})}] &= \mathbb{E}_m\big[f(X_n)\mathbb{E}_{X_{n+1}}[\overline{g(X)}]\big]\\
&= \mathbb{E}_m\big[f(X_n)\overline{\psi(X_1\circ\theta_n)}\big] = \mathbb{E}_m[f(X_n)\overline{P\psi(X_n)}]\\
&= \langle f, P\psi\rangle_{mP^n} = \langle f, P\psi\rangle_m = \langle P^* f, \psi\rangle_m\\
&= \langle P^* f, \psi\rangle_{mP^{n+1}} = \mathbb{E}_m[P^* f(X_{n+1})\overline{\psi(X_{n+1})}]\\
&= \mathbb{E}_m[P^* f(X_{n+1})\overline{g(X\circ\theta_{n+1})}]
\end{aligned}$$
Therefore, $\mathbb{E}_m[f(X_n)|\mathscr{G}_{n+1}] = P^* f(X_{n+1})$. $\square$

- The process $\{\widetilde{Y}_n\}$ with probability transition function P^* is called ***dual process***.
- Since P^* is a contraction, so is P^* and $\|P\|_2 = \|P^*\|_2$. Moreover, if h is ***harmonic***, i.e. $Ph = h$, then $P^*h = h$, and $\{h(X_n), \mathscr{G}_n\}$ is a reversed martingale w.r.t $\mathbb{P}_m$.
- In discrete state space S, the duality relation between P and P^* is equivalent to
$$m(j)p(j,i) = m(i)p^*(i,j), \qquad i,j \in S$$

5. Capacity

The μ***–potential measure*** of A with respect to D is defined as $e^{\mu}_{A,D}(x) := (\mu - L)h^{\mu}_{A,D}(x)$.

- $e^{\mu}_{A,D}$ satisfies
$$e^{\mu}_{A,D}(x) = \begin{cases} (1+\mu)\big(1 - \mathbb{P}_x\big[(1+\mu)^{-\tau_A}; \tau_A < \tau_D\big]\big) & x \in A\\ -(1+\mu)\mathbb{P}_x\big[(1+\mu)^{-\tau_A}; \tau_A < \tau_D\big] & x \in D\\ 0 & x \in \Gamma \end{cases}$$
- Hence $h^{\mu}_{A,D}$ solves the equation
$$\begin{aligned}
(\mu - L)h^{\mu}_{A,D}(x) &= \mathbb{1}_A(x)e^{\mu}_{A,D}(x) \qquad x \in D^c\\
h^{\mu}_{A,D}(x) &= 0 \quad x \in D;
\end{aligned}$$
- Consequently, $h^{\mu}_{A,D}(x) = \big(G_{\mu,D}\mathbb{1}_A e^{\mu}_{A,D}\big)(x)$.
- It is clear that if $\mu \geq 0$, then $e^{\mu}_{A,D}(x)\mathbb{1}_A(x) \geq 0$

Assume P has a (unique) invariant probability measure m. The μ–**capacity** of A with respect to D is defined by

$$\mathrm{cap}^{\mu}(A,D) = \int_A e^{\mu}_{A,D}(x)\, m(dx) \tag{11}$$

- For $\mu = 0$, we have the following probabilistic interpretation:

$$\begin{aligned}\mathrm{cap}(A,D) &= \int_A e_{A,D}(x)\, m(dx)\\ &= \int_A (1-\mathbb{P}_x[\tau_A < \tau_D])\, m(dx) = \mathbb{P}_m[X_0 \in A, \tau_D < \tau_A]\end{aligned}$$

- When $\mu = 0$, observe that

$$1 - h_{A,D}(x) = \mathbb{P}_x[\sigma_D < \sigma_A] = h_{D,A}(x)$$

- Since $L\mathbb{1} = 0$ and $\int Lf\, dm = 0$ for any $f \in L_1$, we have

$$\begin{aligned}\int_A e_{A,D}\, dm = -\int_A e_{D,A}\, dm &= -\int e_{D,A}\, dm + \int_{A^c} e_{D,A}\, dm\\ &= \int_\Gamma e_{D,A}\, dm + \int_D e_{D,A}\, dm = \int_D e_{D,A}\, dm\end{aligned}$$

- Hence, $\mathrm{cap}(A,D) = \mathrm{cap}(D,A)$ or, equivalently,

$$\mathbb{P}_m[X_0 \in A, \tau_D < \tau_A] = \mathbb{P}_m[X_0 \in D, \tau_A < \tau_D]$$

- The ***ergodic flow*** out of A is defined by

$$\Phi(A) := \int_A P\mathbb{1}_{A^c}(x)\, m(dx) = \mathbb{P}_m[X_0 \in A, X_1 \in A^c] \tag{12}$$

- Observe that

$$\begin{aligned}\Phi(A) &= m(A^c) - \int_{A^c} P\mathbb{1}_{A^c}\, dm\\ &= m(A^c) - \big(m(A^c) - \int_{A^c} P\mathbb{1}_A\, dm\big)\\ &= \int_{A^c} P\mathbb{1}_A = \mathbb{P}_m[X_0 \in A^c, X_1 \in A] = \Phi(A^c)\end{aligned}$$

6. Dirichlet form

Suppose, in addition, that P is self–adjoint; that is,

$$\langle Pf, h\rangle_m = \langle f, Ph\rangle_m$$

for all $f \in \mathcal{L}_1(m)$, $h \in \mathcal{L}_\infty(m)$.

- The μ–***Dirichlet form*** $\mathcal{Q}^\mu$ is defined by

$$\mathcal{Q}^{\mu}(f,g) = \int (\mu - L) f(x)\overline{g(x)}\, m(dx)$$

- It can be easily checked that
$$\begin{aligned}\mathcal{Q}^\mu(f,f) &= \frac{1}{2}\int\Big(\int|f(x)-f(y)|^2P(x,dy)+2\mu|f(x)|^2\Big)\,m(dx)\\ &= (\mu+1)\|f\|^2_{L_2(m)}-\langle Pf,f\rangle\geq 0\end{aligned}$$
for all $\mu\geq 0$.

Theorem 6.1. *$h^\mu_{A,D}$ is the unique function that minimizes the form $\mathcal{Q}^\mu(f,f)$ over all $f\in\mathcal{L}_2(m)$ such that $f(x)=\mathbb{1}_A(x)$ for $x\in\Gamma^c=A\cup D$.*

Proof. Denote $h=h^\mu_{A,D}$. Let $g\in\mathcal{L}_2(m)$ real and such that $g(x)=g(x)\mathbb{1}_{A\cup D}(x)$. Then
$$\begin{aligned}\mathcal{Q}^\mu(h+g,h+g)-\mathcal{Q}^\mu(h,h) &= 2\mathcal{Q}^\mu(h,g)+\mathcal{Q}^\mu(g,g)\\ &\geq 2\int_\Gamma(\mu-L)f\cdot g\,dm=0\end{aligned}$$
To show uniqueness, suppose $f\in\mathcal{L}_2$, $f=f\mathbb{1}_A$ and $\mathcal{Q}^\mu(f,f)=\mathcal{Q}^\mu(h,h)$. Then
$$\begin{aligned}\frac{1}{4}\big(\mathcal{Q}^\mu(f+h,f+h)-\mathcal{Q}^\mu(f-h,f-h)\big) &= \frac{1}{2}\big(\mathcal{Q}^\mu(f,f)+\mathcal{Q}^\mu(h,h)\big)\\ &= \mathcal{Q}^\mu(h,h).\end{aligned}$$
Hence, from
$$\mathcal{Q}^\mu(h,h)\leq\mathcal{Q}^\mu(\tfrac{1}{2}(f+h),\tfrac{1}{2}(f+h))=\mathcal{Q}^\mu(h,h)-\mathcal{Q}^\mu(\tfrac{1}{2}(f-h),\tfrac{1}{2}(f-h)),$$
we conclude that $\mathcal{Q}^\mu(\frac{1}{2}(f-h),\frac{1}{2}(f-g))=0$. Therefore, $\|f-h\|_2=0$. □

- The $\mathrm{cap}^\mu(A,D)$ satisfies
$$\begin{aligned}\mathrm{cap}^\mu(A,D) &= \int_A e^\mu_{A,D}\,dm=\int h^\mu_{A,D}(\mu-L)h^\mu_{A,D}\,dm\\ &= \mathcal{Q}^\mu(h^\mu_{A,D},h^\mu_{A,D})\end{aligned}$$
- For $\mu=0$, we have that
$$\begin{aligned}\mathcal{Q}(h_{A,D}) &= \mathrm{cap}(A,D)\\ \mathcal{Q}(\mathbb{1}_A) &= \Phi(A)\end{aligned}$$
- It follows that $\mathrm{cap}(A,D)\leq\Phi(A)$ for any $D\subset A^c$.

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