LIOUVILLE'S THEOREM

Contents

1. Introduction

The purpose of this set of notes is to give a systematic proof of Liouville's Theorem using elements of Calculus.

2. Liouville's Theorem

Consider the autonomous system

$$\dot{x} = f(t, x) \tag{1}$$

where $f : I \times \Omega \subset \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies the smoothness conditions as in the uniqueness and existence theorem. An important property of the solutions to the differential equation (1) is the *flow property*; namely, of $\phi_{t,s}(y) = \phi(t; s, y)$ denotes the solution in t to (1) with initial conditions $\phi(s; s, y) = y$, then

1. $\phi_{t,s} \circ \phi_{s,r}(y) = \phi_{t,r}(y)$ for all r, s, $t \in I$ and $y \in \Omega$.
2. $\phi_{t,t}(y) = y$ for all $t \in I$ and $y \in \Omega$.

For sake of simplicity, we will assume that any solution starting in Ω exists for all times, that is $I = \mathbb{R}$.

Suppose that $D(0) \subset \Omega$ has a finite volume $v(0)$ in $\mathbb{R}^n$; then, the flow $\phi_{0,t}$ transports $D(0)$ to $D(t) = \phi_{t,0}(D(0))$. A problem of interest is to understand how the volume $v(t) = \mathrm{vol}(D(t))$ evolves with $\phi_{t,0}$.

Theorem 2.1. *Consider the differential equation* (1)*, where f is an $\mathbb{R}^n$–valued continuous differentiable function defined on a domain $\mathbb{R} \times \Omega \subset \mathbb{R}^{1+n}$, and let $\phi_{t,s}$ be the flow associated to* (1)*. For any region*

$D(0) \subset \Omega$ with finite volume $v(0)$, let $v(t) = \mathrm{vol}[D(t)] = \mathrm{vol}[\phi_{t,0}(D(0))]$. Then, $v(t)$ satisfies the equation

$$\begin{aligned} \dot{v}(t) &= \int\limits_{D(0)} (\nabla_y \cdot f)(t, \phi(t;0,y)) \det\left[\tfrac{\partial \phi}{\partial y}(t;0,y)\right] dy \\ &= \int\limits_{D(t)} \nabla_y \cdot f(t,y) dy \end{aligned} \tag{2}$$

Corollary 2.2. *If f is incompressible, that is $\nabla_x \cdot f(t,x) = \mathrm{div}_x\, f(t,x) = 0$ for all $t \in \mathbb{R}$ and $x \in \Omega$, then we have that the system $\dot{x} = f(t,x)$ is* volume preserving.

Example 2.1. A Hamiltonian system is a $2n$–dimensional system of equations of the form

$$\begin{aligned} \dot{p}_i &= \frac{\partial H}{\partial q_i} \\ \dot{q}_i &= -\frac{\partial H}{\partial p_i} \end{aligned}$$

with $i = 1, \ldots, n$. $H(\mathbf{p}, \mathbf{q})$ is called Hamiltonian. These systems are volume preserving.

3. Proof of Liouville's theorem

By the change of variables formula for integration, (see for instance [Buc78, pp. 386, 391], [Spi65, Thm. 3.13], [Rud87, Thm. 2.16]) we have that

$$\begin{aligned} v(t) &= \int_{D(t)} dy = \int_{\phi_{t,0} D(0))} dy \\ &= \int_{D(0)} \left| \det\left[\tfrac{\partial \phi_{t,0}}{\partial y}(y)\right] \right| dy \end{aligned} \tag{3}$$

Since $\phi_{t,0}(\cdot)$ is a family of diffeomorphisms and $\phi_{0,0} = Id$, we can ignore the absolute value in (3). Differentiating with respect to t gives

$$\dot{v}(t) = \int_{D(0)} \frac{d}{dt} \det\left[\frac{\partial \phi_{t,0}}{\partial y}(y)\right] dy \tag{4}$$

We will use the following elementary auxiliary Lemma from linear algebra:

Lemma 3.1. *Let $\Delta : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$ be the determinant function, i.e.*

$$\Delta(\alpha_{11}, \ldots, \alpha_{n1}, \ldots, \alpha_{1n}, \ldots, \alpha_{nn})^\top = \det[(\alpha_{ij})]$$

where (α_{ij}) is the $n \times n$–matrix whose ij–th component is α_{ij}. Then,

$$\Delta_\alpha = \frac{\partial \Delta}{\partial \alpha} = (W_{11} \ldots, W_{n1}, \ldots, W_{1n}, \ldots, W_{nn})$$

where W_{ij} is the ij–th cofactor of the matrix (α_{ij}).

Proof. This is an elementary exercise on computing determinants using the cofactor formula [HK71, p. 148]. ☐

Lemma 3.2. *Let* $\phi(t;\mathbf{x})$ *be a solution to the equation* $\dot{\mathbf{x}} = f(t,\mathbf{x})$*, with* $\phi(0;\mathbf{x}) = \mathbf{x}$*. Define the function* W *by*

$$W(t,\mathbf{x}) \;=\; \det\left[\frac{\partial\phi}{\partial\mathbf{x}}(t;\mathbf{x})\right].$$

Then, W *satisfies the differential equation*

$$\dot{W}(t) = W(t)\,(\nabla_{\mathbf{x}}\cdot f)(t,\phi(t;x)); \qquad W(0) = 1, \tag{5}$$

where $(\nabla_{\mathbf{x}}\cdot f)\,(t,\phi(t;\mathbf{x})) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(t,\phi(t;\mathbf{x}))$

Proof. If $\phi(t;\mathbf{x}) = (\phi^1(t;\mathbf{x}),\ldots,\phi^n(t;\mathbf{x}))^\top$, denote by $\phi^i_{x_j}(t;\mathbf{x}) = \frac{\partial\phi^i}{\partial x_j}(t;\mathbf{x})$. Using Lemma 3.1 and the chain rule we have that

$$\begin{aligned}\dot{W} &= \textstyle\sum_i W_{i1}\dot{\phi}^i_{x_1} + \cdots + \sum_i W_{in}\dot{\phi}^i_{x_n} \\ &= \textstyle\sum_{ij} W_{ij}\dot{\phi}^i_{x_j}\end{aligned} \tag{6}$$

where W_{ij} is the ij–th cofactor of the matrix $\left(\phi^i_{x_j}\right)$. Recall from class-notes that $\phi_{\mathbf{x}}(t;\mathbf{x}) = \frac{\partial\phi}{\partial\mathbf{x}}(t;\mathbf{x})$ satisfies the variational equation

$$\begin{aligned}\dot{\phi}_{\mathbf{x}}(t;\mathbf{x}) &= f_{\mathbf{x}}(t,\phi(t;\mathbf{x}))\phi_{\mathbf{x}}(t;\mathbf{x}) \\ \phi_{\mathbf{x}}(0;\mathbf{x}) &= I\end{aligned} \tag{7}$$

substituting (7) on (6) and recalling the fact that the determinant of a matrix that has two identical columns is zero, we obtain

$$\begin{aligned}\dot{W}(t) &= \sum_{ijk} W_{ij}(t) f^i_{x_k}(t,\phi(t;\mathbf{x}))\phi^k_{x_j}(t;\mathbf{x}) \\ &= \sum_{ki}\left(f^i_{x_k}(t,\phi(t;\mathbf{x})\right)\sum_j W_{ij}(t)\phi^k_{x_j}(t;\mathbf{x}) \\ &= \sum_i f^i_{x_i}(t,\phi(t;\mathbf{x})\sum_j W_{ij}(t)\phi^i_{x_j}(t;\mathbf{x}) \\ &= \sum_i f^i_{x_i}(t,\phi(t;\mathbf{x}))W(t) = W(t)\,\left(\nabla_{\mathbf{x}}\cdot f\right)(t,\phi(t;\mathbf{x}))\end{aligned}$$

☐

To conclude the proof of Theorem 2.1, notice that equation (2) follows directly from equation (4) and Lemma 3.2. ☐

Example 3.1. For a non–autonomous linear system

$$\dot{x} = A(t)x \quad x(0) = x_0, \tag{8}$$

where $A(t) \in L(\mathbb{R}^n, \mathbb{R}^n)$ is continuously differentiable in t, we have that

$$W(t) = W(0) \exp\Big(\int_0^t \operatorname{Tr}(A(s))\, ds \Big) \tag{9}$$

where $\operatorname{Tr} A(t) = \sum_j a_{jj}(t)$. This follows from the variational equations and Lemma 3.2. The solution to (9) is known as the Wronskian of (8).

4. Conservation Law.

Theorem 2.1 can be extended to other measures on phase space. Suppose that μ is a measure on $(\Omega, \mathcal{B}(\Omega))$ with smooth density G with respect to Lebesgue measure on Ω; flow ϕ associated to equation (1) satisfies conditions (1) and (2). The flow ϕ_t induces the measure μ_t on phase space $(\Omega, \mathcal{B}(\Omega))$, namely

$$\mu_t(B) := \mu(\phi^{-1}_{t,0}(B)) = \mu(\phi_{0,t}(B)) \tag{10}$$

Furthermore, μ_t also has a density with respect to the Lebesgue measure. This follows from

$$\mu_t(B) = \int_{\phi_{0,t}(B)} G(\mathbf{x}) d\mathbf{x} = \int_B G(\phi_{0,t}(\mathbf{x})) \det \Big[\tfrac{\partial \phi_{0,t}}{\partial x}(\mathbf{x})\Big]\, d\mathbf{x}. \tag{11}$$

So, $w(t,x) = G(\phi_{0,t}(\mathbf{x})) \det \Big[\tfrac{\partial \phi_{0,t}}{\partial x}(\mathbf{x})\Big]$ is a density for μ_t.

The following result generalizes Theorem 2.1 by describing how μ_t changes in time.

Theorem 4.1. *Let ϕ be the flow associated to equation* (1), *and $\mu(d\mathbf{x}) = G(\mathbf{x})d\mathbf{x}$ be a smooth measure on phase space $(\Omega, \mathcal{B}(\Omega))$. Then, the density $w = w(t, \mathbf{x})$ of the induced measure $\mu_t(d\mathbf{x})$ is smooth and satisfies the equation*

$$\frac{\partial w}{\partial t} + \nabla_{\mathbf{x}} \cdot (wf) = 0 \tag{12}$$

Proof. Notice that $\mu(B) = \mu_t(\phi_{t,0}(B))$ for all t and any $B \in \mathcal{B}(\Omega)$; so, $0 = \frac{d}{dt}\mu_t(\phi_{t,0}(B))$. By Lemma 3.2,

$$\begin{aligned} 0 &= \textstyle\int_B \big(\frac{\partial w}{\partial t}(t, \phi_{t,0}(\mathbf{x})) + \nabla_{\mathbf{x}} \cdot (wf)(t, \phi_{t,0}(\mathbf{x}))\big) \det \Big[\frac{\partial \phi_{t,0}}{\partial x}(\mathbf{x})\Big]\, d\mathbf{x} \\ &= \textstyle\int_{\phi_{t,0}(B)} \frac{\partial w}{\partial t}(t, \mathbf{x}) + \nabla_{\mathbf{x}} \cdot (wf)(t, \mathbf{x})\, d\mathbf{x} \end{aligned} \tag{13}$$

for any $B \in \mathcal{B}(\Omega)$. □

Corollary 4.2. *If $w = w(x)$ is a positive stationary solution to equation* (12)*, then $\mu(dx) = w(x)\,dx$ is an invariant measure, i.e., $\mu_t = \mu$ for all t.*

Proof. Suppose w is a positive solution to

$$\nabla_{\mathbf{x}} \cdot (wf) = 0. \tag{14}$$

Then,

$$\mu(\varphi_{t,0}(B)) = \int_B w(\phi_{t,0}(\mathbf{x})) \det \left[\tfrac{\partial \phi_{t,0}}{\partial x}(\mathbf{x})\right] \, dx$$

By Differentiation and Lemma 3.2 we obtain

$$\frac{d}{dt}\mu(\phi_{t,0}(B)) = \int_B \nabla_{\mathbf{x}} \cdot (wf)(t, \phi_{t,0}(\mathbf{x})) \det \left[\tfrac{\partial \phi_{t,0}}{\partial x}(\mathbf{x})\right] \, d\mathbf{x} = 0$$

Hence, $\mu(\varphi_{t,0}(B)) = \mu(B)$ for all t and $B \in \mathcal{B}(\Omega)$ and by taking $\varphi_{0,t}(B)$ in place of B we get $\mu_t = \mu$ for all t. $\square$

Example 4.1. If $f(\mathbf{x})$ is an incompressible field, i.e. $\nabla_{\mathbf{x}} \cdot f = 0$, then taking $w(x) \equiv 1$ we obtain that the Lebesgue measure on $\mathcal{B}(\Omega)$ is invariant under the flow ϕ.

Example 4.2. Consider the Hamiltonian system defined in example 2.1. For any measurable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi \circ H \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^{2n})$, the measure

$$\mu_\varphi(dp, dq) = \varphi(H(p, q))\, dpdq$$

is invariant. Indeed, if $w(p, q) = \varphi(H(p, q))$ and $f = [\partial_q H \ \partial_q H]^T$

$$\begin{aligned}\nabla_{p,q} \cdot (w\, f) = {} & \varphi'(H)\, \partial_p H \cdot \partial_q H + \varphi(H)\partial_{p,q} H \\ & - \varphi'(H)\, \partial_q H \cdot \partial_p H - \varphi(H)\partial_{q,p} H = 0\end{aligned}$$

A special case is

$$\mu(dp, dq) = \frac{e^{-H(p,q)}}{Z}\, dp\, dq$$

where $Z = \int_{\mathbb{R}^{2n}} e^{-H(p,q)}\, dp\, dq$ if the integral exists or 1 otherwise. This measure is called *Gibbs measure.*

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Dept. of Math & Stats. McMaster University, HH 417
E-mail address: `odiaz@math.mcmaster.ca`