LECTURE NOTES ON LÉVY PROCESSES

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1. Examples and definitions

1.1. **Basic results in Probability.** In this section we introduce some general concepts and results from Probability that will be needed in our treatment of the subject. The most important result, theorem 1.16 on characteristic functions of distributions, will be used systematically in this notes. We conclude with the definition of a Lévy processes and three of the most basic examples of such processes, namely Poisson processes, Compound Poisson processes and Brownian motion.

Definition 1.1. A *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ is a triplet of a set Ω, a family $\mathcal{F}$ of subsets of Ω, and a function $\mathbb{P}$ from $\mathcal{F}$ to $\mathbb{R}$ satisfying the following properties:

(1) $\Omega \in \mathcal{F}$
(2) If $A \in \mathcal{F}$ then $\Omega \setminus A \in \mathcal{F}$
(3) If $A_n \in \mathcal{F}$ for $n = 1, \ldots,$ then $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$
(4) $0 \leq \mathbb{P}[A] \leq 1$, $\mathbb{P}[\emptyset] = 0$ and $\mathbb{P}[\Omega] = 1$.
(5) If $A_n \in \mathcal{F}$ for $n = 1, \ldots$ and they are disjoint, then

$$\mathbb{P}\left[\bigcup_{n=1}^{\infty} A_n\right] = \sum_{n=1}^{\infty} \mathbb{P}[A_n]$$

The family of sets $\mathcal{F}$ is usually called σ–algebra of events of Ω. Very often we have a family of indexed σ–algebras, $\{\mathcal{F}_t : t \in [0, \infty)\}$, such that $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ whenever $0 \leq t \leq s$. Such a family is called a filtration and define the *final* σ–algebra to be

$$\mathcal{F}_\infty = \bigvee_t \mathcal{F}_t \stackrel{\text{def}}{=} \sigma\left(\bigcup_t \mathcal{F}_t\right).$$

On any topological space E, the σ–algebra $\mathcal{B}(E)$ generated by the open sets is called the Borel σ–algebra of E.

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Definition 1.2. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we say that a function X from Ω to $\mathbb{R}^d$ is measurable, o simply a random variable, if it is $\mathcal{F}$–measurable, i.e., $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}(\mathbb{R}^d)$. If X is a random variable, the probability measure $\mathbb{P}_X[B] = \mathbb{P}[X^{-1}(B)]$ is called the *law* or *distribution* of X. We write $X \stackrel{d}{=} Y$ to say that the random elements X and Y have the same law or distribution.

Given a collection of measurable spaces $\{(E_t, \mathcal{E}_t) : t \in T\}$ we consider the space of all *choice functions* $f : T \longrightarrow \cup_{t\in T} E_t$ with $f(t) \in E_t$, denoted by $\prod_{t\in T} E_t$, and endow it with the product σ–algebra, i.e., the σ–algebra generated by the projections $p_t : \prod_{t\in T} E_t \longrightarrow E_t$, defined by $p_t(f) = f(t)$.

Lemma 1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Then a function $X : \Omega \longrightarrow \prod_{t\in T} E_t$ is measurable (with respect the product σ–algebra) if and only if for each $t \in T$, $X_t : \Omega \longrightarrow S$, given by $X_t(\omega) = (p_t \circ X)(\omega)$, is $\mathcal{F}$–measurable.*

It is of interest when the spaces $(E_t, \mathcal{E}_t)$ are Polish spaces (separable topological spaces that admit a complete metric) endowed with their Borel σ–algebras. Example of such spaces are separable Banach spaces, $C[0, \infty)$ and $D[0, \infty)$. If $E_t = E$ for all $t \in T$, we simply use the notation E^T to denote the product space. A mapping X with the properties of lemma 1.1 is called an E–valued (random) process on T or a process on T with paths in E.

Let $X_t, t \in T$, be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in Polish spaces $E_t, t \in T$, respectively. The associated *marginal finite dimensional distributions*, given by

$$\mu_{t_1,\dots,t_k} = \mathbb{P} \circ (X_{t_1}, \dots, X_{t_k})^{-1}, \quad t_1, \dots, t_k \in T, k \in \mathbb{N},$$

are consistent in the sense that if $T_1 = \{t_1, \dots, t_m\} \subset T_2 = \{\alpha_1, \dots, \alpha_k\} \subset T$ and μ_1, μ_2 are the respective laws of $(X_{t_1}, \dots, X_{t_m})$, $(X_{\beta_1}, \dots, X_{\beta_k})$, then μ_1 is the image of μ_2 under the projection $\prod_{T_2} E_t \to \prod_{T_1} E_t$. The first of the following two lemmas, which holds true for arbitrary random variables, gives a full description of the product σ–algebra $\sigma(X_t, t \in T)$ in terms of countable ones; The second one shows that the distribution of a process is determined by the set of finite–dimensional distributions.

Lemma 1.2. $\sigma(X_t, t \in T) = \cup\sigma(X_t, t \in J)$ *where the union is taken over all countable $J \subset T$.*

Lemma 1.3. *(finite dimensional distributions) Let X and Y be processes on T with paths in E (not necessarily defined on the same probability space). Then $X \overset{d}{=} Y$ iff*

$$(X_{t_1}, \ldots, X_{t_k}) \overset{d}{=} (Y_{t_1}, \ldots, Y_{t_k}) \quad t_1, \ldots, t_k \in T, k \in \mathbb{N}$$

The next theorem due to Kolmogorov shows that the consistency property described above is sufficient for the realization of such random variables. Also, it warranties the extension to a unique probability measure on product spaces when a system of finite–dimensional distributions with *good compatibility* properties is given.

Theorem 1.4. *(Kolmogorov's extension theorem). Let E_t, $t \in T$, be Polish spaces and $\mu_J, J \subset T$ finite, a consistent family of probability measures on $E_J = \prod_J E_t$ respectively. Then there exists a unique probability measure μ on $E_T = \prod_T E_t$ such that μ_J is the image of μ under the projection $E_J \to E_T$.*

For the proofs of lemmas 1.1 and 1.3 see [6]. For the proofs of lemma 1.2 and of Kolmogorov's extension as stated see [2].

If $T = [0, \infty)$ and X is a random process on T with paths in the Polish space E, then we can associate to it a mapping $\chi : [0, \infty)) \times \Omega \mapsto E$ given by

$$\chi(t, \omega) = X_t(\omega) = (p_t \circ X)(\omega).$$

Consider a filtration $\{\mathcal{F}_t : t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. We say that that the stochastic process X is *adapted* to the filtration $\mathcal{F}_t$ iff for each $t \geq 0$, the mapping $\omega \mapsto \chi(t, \omega) = X_t(\omega)$ is $\mathcal{F}_t$–measurable. Very often we refer to χ as the stochastic process. We say that a process X *measurable* iff $\chi : ([0, \infty) \times \Omega, \mathcal{B}([0, \infty) \otimes \mathcal{F}_\infty)) \longrightarrow (E, \mathcal{B}(E))$ is measurable. The *process X stopped at time t*, denoted by X^t, is given by the mapping $(s, \omega) \mapsto X(s \wedge t, \omega)$. This means that for each $\omega \in \Omega$, the path of $X^t_s(\omega)$ is the constant $X_t(\omega)$ after time t. A process X is *progressively measurable* if for every $t \geq 0$ the stopped process X^t is measurable on $\mathcal{B}([0, \infty) \otimes \mathcal{F}_t)$. Intuitively, this means that any measurable information about the whole path up to time t is contained in $\mathcal{F}_t$.

Proposition 1.5. *(i) A progressively measurable process is adapted.*
(ii) A left– or right–continuous adapted process is progressively measurable.

Proof. (i) $X_t = p_t \circ X$ is the composition of $\omega \mapsto (t, \omega)$ with the mapping χ^t associated to X^t.

(ii) If X is adapted with left–continuous paths, define the processes $X^{(n)}$ by setting

$$X^{(n)}_s(\omega) = X_{k/n}(\omega) \quad \text{for} \;\; k = [n\, s]$$

That each $X^{(n)}$ is progressively measurable follows from

$$((\chi^{(n)})^t)^{-1}(A) = \left(\bigcup_{k=0}^{[nt]} \left[\tfrac{k}{n}, \tfrac{k+1}{n}\right) \times (X_{k/n})^{-1}(A) \right) \\ \bigcup \left(\left[\tfrac{[nt]+1}{n}, \infty\right) \times (X_{k/n})^{-1}(A) \right).$$

Clearly $\chi^{(n)}(s,\omega) \to \chi(s,\omega)$ as $n \to \infty$, which implies that χ is $\mathcal{B}([0,\infty)) \otimes \mathcal{F}_t$–measurable.

Assume now that the paths of X are right–continuous and fixed a time $t \geq 0$. The mapping $\chi^t(s,\omega)$ is clearly the point-wise limit of the functions

$$\chi^{(n)}(s,\omega) = \sum_{k=0}^{\infty} X_{\frac{k+1}{n}\wedge t}(\omega) \cdot \mathbf{1}_{\left(\frac{k}{n}, \frac{k+1}{n}\right]}(s)$$

Each term in the sum above is $\mathcal{B}([0,\infty)) \otimes \mathcal{F}_t$–measurable. $\square$

Given a stochastic process X with values in $\bar{\mathbb{R}}$, we define its *maximal process* X^* by

$$X^*_t(\omega) = \sup_{0 \leq s \leq t} |X^*_s(\omega)|$$

It might be that X^* is not a stochastic process. When X is progressively measurable and the filtration is *universally complete* however, then not only is Z^* a stochastic process but also it is progressively measurable again. See [1] for the details. The case of cadlag processes can be dealt with easily once we notice that in such a case, the supremum over $[0,t]$ can be substituted by the supremum over $\mathbb{Q}_t = (\mathbb{Q} \cap [0,t]) \cup \{t\}$.

Definition 1.3. (Uniform integrability) Let be $(\Omega, \mathcal{F}, \mu)$ a measure space. A family of random vectors $\{X_\alpha : \alpha \in I\}$ is *uniformly integrable* iff:

(1) $\sup\limits_{\alpha \in I} \int_\Omega |X_\alpha|\, d\mu < \infty$

(2) For very $\varepsilon > 0$ there is $\delta > 0$ such that if $\mu(A) < \delta$ then

$$\sup_{\alpha \in I} \int_A |X_\alpha|\, d\mu < \varepsilon$$

It can be easily shown that uniform integrability of the family $\{X_\alpha : \alpha \in I\}$ is equivalent to

$$\lim_{a\to\infty} \sup_{\alpha\in I} \int_{\{|X_\alpha|\geq a\}} |X_\alpha| \, d\mu = 0$$

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an integrable random variable X, we can define a signed (or complex) measure ν by

$$\nu(A) = \int_A X(\omega)\mathbb{P}(d\omega), \quad A \in \mathcal{F}$$

If $\mathcal{C}$ is a sub σ–algebra of Ω contained in $\mathcal{F}$, then the restriction of ν to $\mathcal{C}$, will be absolute continuous with respect to the restriction of $\mathbb{P}$ to $\mathcal{C}$. Thus by Radon–Nikodym's theorem there exists a $\mathbb{P}$–almost surely unique $\mathcal{C}$–measurable function g such that

$$\nu(A) = \int_A g(\omega)\mathbb{P}(d\omega) = \int_A X(\omega)\mathbb{P}(d\omega) \quad \text{for } A \in \mathcal{C}$$

This function g is called *Expectation of X given* $\mathcal{C}$ and is denoted by

$$g = \mathbb{E}\left[X|\mathcal{C}\right]$$

Definition 1.4. Let $(\Omega, \mathcal{F}_\bullet, \mathbb{P})$ a probability space with filtration $\mathcal{F}_\bullet = \{\mathcal{F}_t : t \geq 0\}$. A process M is a martingale with respect to $\mathcal{F}_\bullet$ if

(1) $\mathbb{E}\left[|M_t|\right] < \infty$ for all $t \geq 0$
(2) For $s \leq t$, $\mathbb{E}\left[M_t|\mathcal{F}_s\right] = M_s$

Lemma 1.6. *If $g \in L^1(\mathbb{P})$ and define $M_t = \mathbb{E}\left[g|\mathcal{F}_t\right]$ for $t \geq 0$. Then M is a uniform integrable martingale.*

Proof. The uniform integrability property follows from Chebishev's inequality

$$\mathbb{P}\left[\mathbb{E}[|g|\,|\mathcal{F}_t] \geq a\right] \leq \tfrac{1}{a}\mathbb{E}[|g|].$$

and from Jensen's inequality which gives the estimate

$$\mathbb{E}\left[|M_t|\mathbf{1}_{[|M_t|\geq a]}\right] \leq \mathbb{E}\left[|g|\mathbf{1}_{[\mathbb{E}[|g|\,|\mathcal{F}_t]\geq a]}\right]$$

Combining both both estimates we obtain

$$\lim_{a\to\infty} \sup_{t\geq 0} \mathbb{E}\left[|M_t|\mathbf{1}_{[|M_t|\geq a]}\right] = 0$$

The martingale property follows from the fact that if $\mathcal{C} \subset \mathcal{B} \subset \mathcal{F}$ are σ–algebras and g is $\mathcal{F}$–measurable then

$$\mathbb{E}[g|\mathcal{C}] = \mathbb{E}\left[\mathbb{E}[g|\mathcal{B}]|\mathcal{C}\right]$$

which is easy to check from the definition of conditional expectation. □

Definition 1.5. Let be $(\Omega, \mathcal{F}_\bullet, \mathbb{P})$ a probability space with filtration $\mathcal{F}_\bullet = \{\mathcal{F}_t : \mathcal{F}_t \subset \mathcal{F}\}$. A function $T : \Omega \mapsto [0, \infty]$ is called a stopping time with respect to $\mathcal{F}_t$ if for all $t \geq 0$ $[T \leq t] \in \mathcal{F}_t$.

The σ–algebra $\mathcal{F}_T$ generated by T is given by

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap [T \leq t] \in \mathcal{F}_t\}$$

It is obvious that if S and T are stopping times such that $S \leq T$, then $\mathcal{F}_S \subset \mathcal{F}_T$. It is easily proved that if M is a martingale with respect $\mathcal{F}_\bullet$, and T is a stopping time taking a countable number of values, then M_T is $\mathcal{F}_T$ measurable.

Lemma 1.7. *Let M be a martingale, S and T be stopping times, both taking a finite number of values and such that $S \leq T$. Then*

(i) for any $u \geq T$

$$\mathbb{E}\left[M_u | \mathcal{F}_T\right] = M_T$$

(ii)

$$\mathbb{E}\left[M_T | \mathcal{F}_S\right] = M_S$$

Proof. Let $0 \leq t_1 < \ldots < t_N = t$ be an ordering of the values of T. For $A \in \mathcal{F}_T$ we have that

$$\begin{aligned} A \cap [T = t_1] &= A \cap [T \leq t_1] \in \mathcal{F}_{t_1} \\ A \cap [T = t_j] &= A \cap [T \leq t_j] \cap [T \leq t_{j-1}]^c \in \mathcal{F}_{t_j} \quad \text{for} \quad j > 1 \end{aligned}$$

Therefore for $u \geq t$

$$\begin{aligned} \mathbb{E}\left[M_T \mathbf{1}_A\right] &= \mathbb{E}\left[\sum_{j=1}^N M_T \mathbf{1}_{[T=t_j]\cap A}\right] = \sum_{j=1}^N \mathbb{E}\left[M_{t_j} \mathbf{1}_{[T=t_j]\cap A}\right] \\ &= \sum_{j=1}^N \mathbb{E}\left[\mathbf{1}_{[T=t_j]\cap A} \mathbb{E}\left[M_u | \mathcal{F}_{t_j}\right]\right] = \sum_{j=1}^N \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{[T=t_j]\cap A} M_u \middle| \mathcal{F}_{t_j}\right]\right] \\ &= \sum_{j=1}^N \mathbb{E}\left[\mathbf{1}_{[T=t_j]} \mathbf{1}_A M_u\right] = \mathbb{E}[M_u \mathbf{1}_A] \end{aligned}$$

Part *(i)* follows from here. Now, if $S \leq T$ then

$$M_S = \mathbb{E}\left[M_u | \mathcal{F}_S\right] = \mathbb{E}\left[\mathbb{E}\left[M_u | \mathcal{F}_T\right] \middle| \mathcal{F}_S\right] = \mathbb{E}\left[M_T | \mathcal{F}_S\right]$$

This shows *(ii)*. □

Lemma 1.8. *(Doob's Maximal lemma) Let M be a martingale and define*

$$M_t^{\mathbb{Q}} = \sup_{s \in \mathbb{Q}_t} |M_s|$$

Then at any instant $t \geq 0$ and for any $\lambda > 0$

$$\mathbb{P}\left[M_t^{\mathbb{Q}} > \lambda\right] \leq \frac{1}{\lambda} \int_{\left[M_t^{\mathbb{Q}} > \lambda\right]} |M_t| \, d\mathbb{P} \leq \frac{1}{\lambda} \mathbb{E}[|M_t|]$$

Proof. Let $\mathcal{S} = \{s_0 < s_1 < \ldots < s_N\}$ be a finite set of numbers in $\mathbb{Q}_t$ and let $u > t$. Set

$$M_S = \sup_{s \in \mathcal{S}} |M_s| \quad \text{and} \quad U = \inf\{s \in \mathcal{S} : |M_s| > \lambda\} \wedge u$$

Clearly U is a stopping time and $|M_U| = |M_{U \wedge t}| > \lambda$ on the set

$$[U < u] = [U \leq t] = \left[M^{\mathcal{S}} > \lambda\right] \subset \left[M_t^{\mathbb{Q}} > \lambda\right] \in \mathcal{F}_t$$

Therefore

$$\mathbf{1}_{[M^{\mathcal{S}} > \lambda]} \leq \frac{|M_U|}{\lambda} \cdot \mathbf{1}_{[U \leq t]}$$

and the function on the right–hand side is $\mathcal{F}_t$–measurable. Now, we apply expectation on both sides and by Lemma 1.7 we get

$$\begin{aligned}
\mathbb{P}\left[M^S > \lambda\right] &\leq \lambda^{-1} \int_{[U \leq t]} |M_U| \, d\mathbb{P} \leq \lambda^{-1} \int_{[U \leq t]} \mathbb{E}\left[|M_u| \middle| \mathcal{F}_U\right] \, d\mathbb{P} \\
&= \lambda^{-1} \int_{[M^S > \lambda]} |M_u| \, d\mathbb{P} \leq \lambda^{-1} \int_{\left[M_t^{\mathbb{Q}} > \lambda\right]} |M_u| \, d\mathbb{P} \\
&\leq \lambda^{-1} \int_{\left[M_t^{\mathbb{Q}} > \lambda\right]} |M_t| \, d\mathbb{P} \leq \lambda^{-1} \mathbb{E}[|M_t|]
\end{aligned}$$

We take now the supremum over all finite subsets of $\mathbb{Q}_t$ and Doob's inequality follows. □

Given two process X_t and Y_t, we say that one is modification of the other if

$$\mathbb{P}[X_t \neq Y_t] = 0 \quad \text{for each } t \geq 0$$

Definition 1.6. A stochastic processes $\{X_t : t \geq 0\}$ with paths in $\mathbb{R}^d$ is continuous in probability, if for every $t \geq 0$ and $\varepsilon > 0$

$$\lim_{s \to t} \mathbb{P}\left[|X_s - X_t| > \varepsilon\right] = 0$$

Analogously, we can define right–continuity in probability by replacing $s \to t$ by $s \downarrow t$.

Lemma 1.9. *A continuous process in probability $\{X_t : t \geq 0\}$ is uniformly continuous in probability on any finite interval $[0, t_0]$, i.e., for every $\varepsilon > 0$ and $\eta > 0$, there is $\delta > 0$ such that if $s, t \in [0, t_0]$ and $|t - s| < \delta$, then $\mathbb{P}[|X_t - X_s| > \varepsilon] < \eta$.*

Proof. Fix ε and η positive. For each $t \in [0, t_0]$ there is δ_t such that $\mathbb{P}\left[|X_t - X_s| > \frac{\varepsilon}{2}\right] < \frac{\eta}{2}$ whenever $|t - s| < \delta_t$. The intervals $I_t = (t - \frac{\delta_t}{2}, t + \frac{\delta_t}{2})$ form an open cover of $[0, t_0]$. Take a finite sub-cover $\{I_{t_j} : j = 1, \dots, n\}$ and let

$$\delta = \min_{1 \leq j \leq n} \frac{\delta_{t_j}}{2}.$$

It follows easily that if $t, s \in [0, t_0]$ and $|t - s| < \delta$ then

$$\mathbb{P}[|X_t - X_s| > \varepsilon] < \eta$$

This concludes the proof. □

The idea of of up-crossings, due to Doob, will be at handy in order to show the following result

Theorem 1.10. *(Regularity of paths, Doob) A right–continuous in probability martingale M has a modification M' which has right–continuous paths with left limits (rcll or cadlag), and a modification M'' which has left–continuous paths with right limits (lcrl or caglad).*

Proof. Let M_t be a real valued martingale right–continuous in probability. Consider two numbers $a < b$, an instant $0 < u < \infty$, and a finite set

$$\mathcal{S} = \{s_0 < s_1 \ldots < s_N\}$$

of numbers strictly less than u. We define the following random times:

$$\begin{aligned} T_0 &= \inf\{s \in \mathcal{S} : M_s < a\} \wedge u \\ T_{2k+1} &= \inf\{s \in \mathcal{S} : s > T_{2k}, M_s > b\} \wedge u \\ T_{2k} &= \inf\{s \in \mathcal{S} : s > T_{2k-1}, M_s < a\} \wedge u. \end{aligned}$$

Clearly, each T_n is a stopping time. We say that the path $M_t(\omega)$ *up-crosses* the interval $[a, b]$ on $\mathcal{S}$ if there are points $s < t$ in $\mathcal{S}$ with $M_s(\omega) < a$ and $M_t(\omega) > b$. We will be interested in estimating the number $U^{[a,b]}_{\mathcal{S}}$ of up–crossings that the path of M has on $\mathcal{S}$. If the path of $M_t(\omega)$ ups-crosses the interval $[a, b]$ at least n times, it means that there are n pairs $s_1 < t_1 < s_2 < t_2 < \ldots < s_n < t_n$ in $\mathcal{S}$ with $M_{s_j} < a$ and $M_{t_j} > b$. This implies $T_{2n-1}(\omega) < u$ and vice versa. In other words

$$\left[U^{[a,b]}_{\mathcal{S}} \geq n\right] = [T_{2n-1} < u] \in \mathcal{F}_u$$

It follows that

$$\mathbf{1}_{\left[U_{\mathcal{S}}^{[a,b]}\leq n\right]} \leq \frac{1}{n(b-a)}\left(\sum_{k=0}^{\infty}(M_{T_{2k+1}} - M_{T_{2k}}) + |M_u - a|\right) \tag{1}$$

To see this, notice that the sum in the right–hand side of (1) has at most $2N$ terms, each of which contributes by a number no less than $b-a$, with the exception perhaps of the last term. That only happens when $T_{2k} < s_N$, which means that $M_{T_{2k}}(\omega) < a$, and when $T_{2k+1} = u$. In that case, the last term is given by $M_u(\omega) - M_{T_{2k}}(\omega)$, which might be negative. If so, it is certainly not smaller that $M_u(\omega) - a$. That is why we need to add term $|M_u - a|$.
Lemma 1.7 implies that

$$\mathbb{P}\left[U_{\mathcal{S}}^{[a,b]} \geq n\right] \leq \frac{1}{n(b-a)}\mathbb{E}\left[|M_u - a|\right]$$

Let $\mathbb{Q}_-^u = \mathbb{Q}\bigcap[0,u)$. Taking supremum over all finite sets $\mathcal{S} \subset \mathbb{Q}_-^u$ and then union over all $a < b$ in $\mathbb{Q}$ we obtain that the set

$$\bigcup_{a,b\in\mathbb{Q},a<b}\left[U_{\mathbb{Q}_-^u}^{[a,b]} = \infty\right] \tag{2}$$

is a null set. Similarly we get that the set

$$Osc = \bigcup_{n\in\mathbb{N}}\bigcup_{a,b\in\mathbb{Q},a<b}\left[U_{\mathbb{Q}_-^n}^{[a,b]} = \infty\right]$$

is null. By Doob's Maximal lemma we have that

$$P = \bigcup_{n\in\mathbb{N}}\left[M_n^{\mathbb{Q}} = \infty\right]$$

is null as well. Let us denote by Ω_0 the complement of $Osc\bigcup P$ and define

$$M_t'(\omega) = \begin{cases} \lim\limits_{\mathbb{Q}\ni q\downarrow t} M_q(\omega) & \text{for } \omega\in\Omega_0 \\ 0 & \text{for } \omega \in Osc\bigcup P \end{cases}$$

$$M_t''(\omega) = \begin{cases} M_0 & \text{for } \quad \omega\in\Omega_0 \quad \text{and} \quad t=0 \\ \lim\limits_{\mathbb{Q}\ni q\uparrow t} M_q(\omega) & \text{for } \omega\in\Omega_0 \\ 0 & \text{for } \omega \in Osc\bigcup P \end{cases}$$

The processes M' and M'' are well defined since for each $t \geq 0$, the limits above exists as a real numbers. Moreover, the process M' is cadlag, while the process M'' is caglad. The continuity in probability implies that for each $t \geq 0$, $M_t = M_t' = M_t''$ a.s. Thus M_t' and M_t'' are modifications of M_t. It is clear that M'' is also a martingale with respect to $\mathcal{F}_\bullet$. On the other hand, both M_t and M'_t are measurable with respect to $\mathcal{F}^+_t = \bigcap_{\mathbb{Q}\ni q>t}\mathcal{F}_q$. Therefore, if $\mathcal{F}_\bullet$ is right–continuous, i.e. $\mathcal{F}_t = \mathcal{F}^+_t$ for all $t \geq 0$, then M' is a martingale with respect to $\mathcal{F}_\bullet$. □

Corollary 1.11. *Let M be a uniform integrable continuous in probability martingale with respect to the filtration $\mathcal{F}_\bullet$. Then there exits a $\mathcal{F}_\infty$–measurable function M_∞ such that $M_t = \mathbb{E}[M_\infty|\mathcal{F}_t]$ for all $t \geq 0$. Furthermore, $M_t \to \mathbb{E}[M_\infty|\mathcal{F}_t]$ a.s and in $L_1(\Omega)$ norm.*

Proof. Let $\mathbb{Q}^\infty_- = \mathbb{Q}\cap[0,\infty)$ and $M^{\mathbb{Q}}_\infty = \sup\limits_{q\in\mathbb{Q}^\infty_-}|M_q|$. Slight changes in the proofs of the Doob's maximal lemma and Doob's regularity theorem give:

$$\begin{aligned}\mathbb{P}[M^{\mathbb{Q}}_\infty = \infty] &= 0\\ \mathbb{P}[U_{\mathbb{Q}^\infty_-} = \infty] &= 0\end{aligned}$$

Let us denote the set $[M^{\mathbb{Q}}_\infty = \infty] \cup [U_{\mathbb{Q}^\infty_-} = \infty]$ by Ω'_0. We define M'' as in Theorem 1.10 but with Ω'_0 instead. Not only is M'' well defined and $M''_t = M_t$ a.s. at any finite time, but also it can be extended to infinity:

$$M''_\infty = \lim_{\mathbb{Q}^\infty_- = \mathbb{Q}\ni q\uparrow\infty} M''_q \overset{\text{a.s.}}{=} \lim_{\mathbb{Q}^\infty_- = \mathbb{Q}\ni q\uparrow\infty} M_q$$

Let $M_\infty = \mathbb{E}[M''_\infty|\mathcal{F}_\infty]$. Fatou's lemma implies that M''_∞ is integrable:

$$\mathbb{E}[|M_\infty|] \leq \liminf_{\mathbb{Q}^\infty_- = \mathbb{Q}\ni q\uparrow\infty} \mathbb{E}[|M_q|] \leq \sup_{t\geq 0}\mathbb{E}[|M_t|]$$

Uniform integrability implies that $\mathbb{E}[|M_\infty - M_q|] \to 0$ as $\mathbb{Q} \ni q \uparrow \infty$ since

$$\mathbb{E}\left[|M_\infty - M_q|\right] = \mathbb{E}\left[|M_\infty - M_q|\mathbf{1}_{[|M_\infty - M_q|\leq\varepsilon]}\right] + \mathbb{E}\left[|M_\infty - M_q|\mathbf{1}_{[|M_\infty - M_q|>\varepsilon]}\right]$$

The conclusion follows from the estimate

$$\mathbb{E}\left[|\mathbb{E}[M_\infty|\mathcal{F}_t] - M_t|\right] \leq \mathbb{E}[|M_\infty - M_q|]$$

for all $\mathbb{Q}\cap[t,\infty)\ni q$. Then we pass to the limit. □

Corollary 1.12. *Let M be a continuous in probability martingale with respect to the filtration $\mathcal{F}_\bullet$, and Ω_0 as in Theorem 1.10. Then for each $t > 0$*

$$\mathbb{P}\left[\Omega_0 \cap [|M_t - M_{t-}| > 0]\right] = 0.$$

Proof. By Theorem 1.10 we have that for each $t \geq 0$ $M_t = M''_t$ a.s. Therefore in Ω_0 we have

$$\begin{aligned} M_{t-} = \lim_{s \uparrow t} M_s &= \lim_{\mathbb{Q} \ni s \uparrow t} M_s \\ &= M''_t \overset{\text{a.s.}}{=} M_t \end{aligned}$$

Thus a.s. there are no fixed jumps. □

We introduce now the concept of independence of sets and of random variables.

Definition 1.7. Let be $(\Omega, \mathcal{F}, \mathbb{P})$ be a probabilistic space. The sets $\{A_\alpha : \alpha \in I\}$ are said to be independent if $\mathbb{P}\left[\bigcap_{\alpha \in J} A_\alpha\right] = \prod_{\alpha \in J} \mathbb{P}[A_\alpha]$ for any finite subset J of I. The families in $\{\mathcal{C}_\beta : \beta \in K\}$, where $\mathcal{C}_\beta \subset \mathcal{F}$, are said to be independent, if the sets in any family of the form $\{A_\beta : A_\beta \in \mathcal{C}_\beta\}$ are independent. We extend the notion of independence to random elements by applying the concept to the σ-algebras generated by them.

For a sequence of events $\{A_n\}$, the upper limit event and the lower limit event are defined by

$$\limsup_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \quad \text{and} \quad \liminf_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

An important result related to independence is

Proposition 1.13. *(Borel–Cantelli)*

it (i) *If* $\sum_{n=1}^{\infty} \mathbb{P}[A_n] < \infty$, *then* $\mathbb{P}[\limsup_{n \to \infty} A_n] = 0$.
it (ii) *If* $\{A_n : n = 1, \dots\}$ *is independent and* $\sum_{n=1}^{\infty} \mathbb{P}[A_n] = \infty$, *then we have* $\mathbb{P}[\limsup_{n \to \infty} A_n] = 1$

A σ–algebra $\mathcal{F}$ is said to be $\mathbb{P}$–trivial if $\mathbb{P}[A] = 0$ or $\mathbb{P}[A] = 1$ for every $A \in \mathcal{F}$. Given a sequence of σ–algebras $\{\mathcal{F}_n\}$, we may introduce the associated tail σ–algebra

$$\mathcal{T} = \bigcap_n \bigvee_{k \geq n} \mathcal{F}_k = \bigcap_n \sigma\left(\mathcal{F}_k : k \geq n\right)$$

Theorem 1.14. *(Kolmogorov's 0–1 law) Let* $\{\mathcal{F}_n\}$ *independent* σ*–algebras. Then the tail* σ*–algebra* $\mathcal{T}$ *is* $\mathbb{P}$*–trivial.*

Proofs of the last results can be found in [2] or [6].

Definition 1.8. The characteristic function $\widehat{\mu}(z)$ of a probability measure μ on $\mathbb{R}^d$ is

$$\widehat{\mu}(z) = \int\limits_{\mathbb{R}^d} e^{i\langle z,x\rangle}\mu(dx), \qquad z \in \mathbb{R}^d$$

Definition 1.9. Consider the measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$

it (i) A collection $\mathcal{P}$ of distributions is *uniformly tight* if for any $\epsilon > 0$ there is a compact set K such that $\inf\limits_{\mu\in\mathcal{P}} \mu(K) \geq 1 - \epsilon$

it (ii) A sequence of distributions μ_n, $n = 1, 2, \dots$ converges in law to a distribution μ, which we will denote by $\mu_n \xrightarrow{d} \mu$ as $n \to \infty$, if for every bounded continuous function f,

$$\int\limits_{\mathbb{R}^d} f(x)\mu_n(dx) \to \int\limits_{\mathbb{R}^d} f(x)\mu(dx) \quad \text{as } n \to \infty$$

When μ and μ_n are bounded measures, the convergence $\mu_n \xrightarrow{d} \mu$ is defined in the same way.

(iii) A Borel set B is a μ–continuity set if $\mu(\partial B) = 0$.

The following theorem contains a fundamental set of results in convergence in law

Theorem 1.15. *If $p \in \mathbb{R}^d$, X, $\{X_n\}$ are random variables, and μ and $\{\mu_n\}$ are distributions on $\mathbb{R}^d$, then*

(a) If $\mu_n \xrightarrow{d} \mu$ then $\{\mu_n\}$ is uniformly tight.
(b) If $\{\mu_n\}$ is uniformly tight, there is a subsequence $\{\mu_{n_k}\}$ that converges in law to some distribution $\mathbb{P}$.
(c) If X_n converges in μ–probability to X, i.e., $\lim_n \mu\left[|X_n - X| > \varepsilon\right] = 0$ for any $\varepsilon > 0$, then $X_n \xrightarrow{d} X$.
(d) X_n converges in μ–probability to p iff $X_n \xrightarrow{d} \delta_p$
(e) The following are equivalent
  (i) $\mu_n \xrightarrow{d} \mu$
  (ii) For all open U, $\liminf_n \mu_n(U) \geq \mu(U)$
  (iii) For all closed F, $\limsup_n \mu_n(F) \leq \mu(F)$
  (iv) or all μ–continuity sets B, $\lim_n \mu_n(B) = \mu(B)$

For a proof of this, see [3].

Definition 1.10. The convolution μ of two finite measures μ_1 and μ_2 on $\mathbb{R}^d$, denoted by $\mu = \mu_1 * \mu_2$, is a measure defined by

$$\mu(B) = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \mathbf{1}_B(x+y)\mu_1(dx)\mu_2(dx), \quad B \in \mathcal{B}(\mathbb{R}^d)$$

The convolution operation is commutative and associative. If X_1 and X_2 are independent random variables on $\mathbb{R}^d$ with distributions μ_1 and μ_2, respectively, then $X_1 + X_2$ has distribution $\mu_1 * \mu_2$.

In the following theorem, we state some results that summarize some of the most important properties of the characteristic function. First, we introduce the following terminology: for a probability measure μ on $\mathbb{R}^d$, $\tilde{\mu}$ is its dual if $\tilde{\mu}(B) = \mu(-B)$; $\mu^{\#}$ is its *symmetrization* if $\mu^{\#} = \mu * \mu$. When $d = 1$, the dual is usually called the *reflection*. If μ agrees with its dual, then it is called *symmetric*.

Theorem 1.16. *Let μ,$\{\mu_n : n \in \mathbb{N}\}$ be distributions on $\mathbb{R}^d$.*

(i) (Bochner's theorem) We have that $\widehat{\mu}(0) = 1$, $|\widehat{\mu}(z)| \leq 1$, $\widehat{\mu}(z) = \overline{\widehat{\mu}(-z)}$, $\widehat{\mu}(z)$ is uniformly continuous, and nonnegative–definite in the sense that for each $n = 1, 2, \ldots$,

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \widehat{\mu}(z_j - z_k)\xi_j \bar{\xi_k} \geq 0 \quad \textit{for } z_1, \ldots, z_n \in \mathbb{R}^d,\ \xi_1, \ldots, \xi_n \in \mathbb{C}$$

Conversely, if a complex–valued function $\varphi(z)$ on $\mathbb{R}^d$ with $\varphi(0) = 1$, uniformly continuous and $\varphi(z) = \overline{\varphi(-z)}$, then φ is the characteristic function of some distribution on $\mathbb{R}^d$.

(ii) If $\widehat{\mu}_1(z) = \widehat{\mu}_2(z)$ for all $z \in \mathbb{R}^d$, then $\mu_1 = \mu_2$.

(iii) If $\mu = \mu_1 * \mu_2$, then $\widehat{\mu}(z) = \widehat{\mu}_1(z)\widehat{\mu}_2(z)$. If X_1 and X_2 are independent random variables on $\mathbb{R}^d$, then $\widehat{\mathbb{P}}_{X_1+X_2}(z) = \widehat{\mathbb{P}}_{X_1}(z)\widehat{\mathbb{P}}_{X_2}(z)$

(iv) Let $X = (X_j)_{j=1,\ldots,n}$ be an $\mathbb{R}^{nd}$–valued variable, where X_j is an $\mathbb{R}^d$–valued variable for each $j = 1, \ldots, n$. Then $\{X_j : j = 1, \ldots, n\}$ are independent iff

$$\widehat{\mathbb{P}}_X(z) = \prod_{j=1}^{n} \widehat{\mathbb{P}}_j(z_j) \qquad \textit{for } z = (z_j)_{j=1,\ldots,n}$$

(v) If $\tilde{\mu}$ and $\mu^{\#}$ are the dual and symmetrazation of μ respectively, then $\widehat{\tilde{\mu}}(z) = \widehat{\mu}(-z)$ and $\widehat{\mu^{\#}}(z) = |\widehat{\mu}(z)|^2$

- *(vi) If $\mu_n \xrightarrow{d} \mu$, then $\widehat{\mu}_n(z) \to \widehat{\mu}(z)$ uniformly on compact sets.*

(vii) If $\widehat{\mu}_n(z) \to \widehat{\mu}(z)$ for every z, then $\mu_n \xrightarrow{d} \mu$

(viii) (Lévy continuity) If $\widehat{\mu}_n(z)$ converges to a function $\varphi(z)$ pointwise and $\varphi(z)$ is continuous at $z = 0$, then $\varphi(z)$ is the characteristic function of some distribution.

(ix) Let n be a positive integer. If $\int_{\mathbb{R}^d} |x|^n \mu(dx) < \infty$, then $\widehat{\mu}(z)$ is a C^n–class function, and for any nonnegative numbers $n_1, \ldots, n_d$

such that $n_1 + \ldots + n_d \le n$,

$$\int\limits_{\mathbb{R}^d} x_1^{n_1} \cdots x_d^{n_d} \mu(dx) = \Big[\Big(\tfrac{1}{i}\tfrac{\partial}{\partial z_1}\Big)^{n_1} \cdots \Big(\tfrac{1}{i}\tfrac{\partial}{\partial z_d}\Big)^{n_d} \widehat{\mu}(z)\Big]_{z=0}$$

(x) *Let* n *be a positive even integer. If* $\widehat{\mu}(z)$ *is of class* C^n *in a neighborhood of the origin, then* $\int_{\mathbb{R}^d} |x|^n \mu(dx) < \infty$.

(xi) *If* $\int_{\mathbb{R}^d} |\widehat{\mu}(z)|\, dz < \infty$, *then* μ *is absolutely continuous with respect to Lebesgue measure, has a bounded continuous density* $g(x)$, *and*

$$g(x) = \tfrac{1}{(2\pi)^{d/2}} \int\limits_{\mathbb{R}^d} e^{-i\langle x,z\rangle} \widehat{\mu}(z)\, dz$$

Proofs can be found in [5], [3] and many other books.

Corollary 1.17. *(Inheritance of independence) Suppose that for each* $j = 1, \ldots, k$, $X_{j,n} \to X_j$ *in probability as* $n \to \infty$. *If the family* $\{X_{j,n} : j = 1, \ldots, k\}$ *is independent for each* n, *then the family* $\{X_j : j = 1, \ldots, k\}$ *is independent.*

The following concept and results will be useful in the characterization of Lévy processes

Definition 1.11. A triangular or null array is an indexed collection of random variables or vectors $\xi_{nj}, n \in \mathbb{N}, 1 \le j \le m_n$, such that for each n, the variables ξ_{nj} are independent and satisfy

$$\sup_j \mathbb{E}[|\xi_{nj}| \wedge 1] \to 0 \quad \text{as } n \to \infty \tag{3}$$

It is an easy exercise to show that condition 3 is equivalent to

$$\sup_j \mathbb{P}[|\xi_{nj}| > \varepsilon] \to 0 \quad \text{as } n \to 0 \quad \text{for all } \varepsilon > 0. \tag{4}$$

Theorem 1.18. *(Poisson convergence) Let* (ξ_{nj}) *be a null array of* $\mathbb{Z}_+$*–valued random variables, and let* ξ *be Poisson distributed with mean* c. *Then* $\sum_j \xi_{nj} \xrightarrow{d} \xi$ *iff these conditions hold:*

(i) $\sum_j \mathbb{P}[\xi_{nj} > 1] \to 0$;

(ii) $\sum_j \mathbb{P}[\xi_{nj} = 1] \to c$.

Moreover, (i) is equivalent to $\sup_j \xi_{nj} \vee 1 \xrightarrow{\mathbb{P}} 1$. *If* $\sum_j \xi_{nj}$ *converges in distribution, then (i) holds iff the limit is Poisson.*

Theorem 1.19. *(Gaussian convergence, Feller, Lévy) Let (ξ_{nj}) be a null array of random variables, and let ξ be $N(b,c)$ distributed for some constants $b \in \mathbb{R}$ and $c > 0$. Then $\sum_j \xi_{nj} \xrightarrow{d} \xi$ iff the following conditions hold:*

(i) $\sum_j \mathbb{P}[|\xi_{nj}| > \varepsilon] \to 0$ *for all* $\varepsilon > 0$;
(ii) $\sum_j \mathbb{E}[\xi_{nj}; |\xi_{nj}| \le 1] \to b$;
(iii) $\sum_j var[\xi_{nj}; |\xi_{nj}| \le 1] \to c$.

Moreover, (i) is equivalent to

$$\sup_j |\xi_{nj}| \xrightarrow{\mathbb{P}} 0$$

almost surely. If $\sum_j \xi_{nj}$ converges in distribution, then (i) holds iff the limit is Gaussian.

For proofs of theorems (1.18) and (1.19) look [6] chapter 4.

Definition 1.12. A stochastic processes $\{X_t : t \ge 0\}$ on $\mathbb{R}^d$ is a Lévy process if the following conditions are satisfied:

(i) For any choice of $n \ge 1$ and $0 \le t_0 < t_1 < \ldots < t_n$ the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent
(ii) $X_0 = 0$ a.s.
(iii) The distribution of $X_{s+t} - X_s$ does not depend on s
(iv) It is stochastically continuous
(v) There is $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}[\Omega_0] = 1$, such that for every $\omega \in \Omega_0$, $X_t(\omega)$ is right-continuous in $t \ge 0$ and has left limits in $t > 0$

Remark 1.1. (1)
(2) Dropping (v) and assuming (i)–(iv) we have a Lévy process in law.
(3) A process satisfying (i), (ii), (iv) and (v) is called *additive process.*
(4) If a process satisfies (i), (ii) and (iv) is called *additive process in law.*
(5) An additive process in law can be *modified* in a set of measure 0 in such a way that the modification is an additive process. See theorem 2.21.

We present some general theorems that we will be needed in the analytic proof of the Itô–Lévy decomposition. For $0 < t < \infty$ let

$D([0,t],\mathbb{R}^d)$ be the set of functions $\xi(s)$ from $[0,t]$ to $\mathbb{R}^d$ with are right continuous for $s\in[0,t)$ and with left limits for $s\in(0,t]$. We introduce the norm $\|\xi\|_t=\sup_{s\in[0,t]}|\xi(s)|$. It is well known that with this norm, the space $D([0,t],\mathbb{R}^d)$ becomes a complete norms space.

Lemma 1.20. *Fix $t>0$. Let $\{Z_j(s):s\in[0,t]\}$, $j=1,2,\dots,$ be independent stochastic processes and $S_0(s)=0$, $S_n(s)=\sum_{j=1}^n Z_j(s)$ for $n=1,2,\dots$. Suppose that for each j, the sample function $Z_j(s)$ belong to $D([0,t],\mathbb{R}^d)$ a.s. Then, for any $\varepsilon>0$ and $n\in\mathbb{N}$,*

$$\mathbb{P}\left[\max_{1\le j\le n}\|S_j\|_t>3\varepsilon\right]\le 3\max_{1\le j\le n}\mathbb{P}\left[\|S_j\|_t>\varepsilon\right]. \tag{5}$$

Proof. Let $M_0=0$ and $M_k=\max_{1\le j\le k}\|S_j\|_t$ for $k\ge1$. Let a, $b>0$, and $A_k=[M_{k-1}\le a+b<\|S_k\|_t]$ for $k\ge1$. Then $A_1,\dots,A_n$ are disjoint and $[M_n>a+b]=\bigcup_{k=1}^n A_k$. Thus

$$\mathbb{P}\left[\|S_n\|_t>a\right]\ge\sum_{k=1}^n\mathbb{P}\left[A_k\cap(\|S_n\|_t>a)\right]\ge\sum_{k=1}^n\mathbb{P}\left[A_k\cap(\|S_n-S_k\|_t\le b)\right]$$

$$=\sum_{k=1}^n\mathbb{P}[A_k]\mathbb{P}\left[\|S_n-S_k\|_t\le b\right]\ge\mathbb{P}[M_n>a+b]\min_{1\le k\le n}\mathbb{P}\left[\|S_n-S_k\|_t\le b\right]$$

Taking $a=\varepsilon$ and $b=2\varepsilon$. Then

$$\begin{aligned}\mathbb{P}[\|S_n\|_t>\varepsilon] &\ge \mathbb{P}[M_n>3\varepsilon]\left(1-\max_{1\le k\le n}\mathbb{P}[\|S_n-S_k\|_t>2\varepsilon]\right)\\ &\ge \mathbb{P}[M_n>3\varepsilon]\left(1-2\max_{1\le k\le n}\mathbb{P}[\|S_k\|_t>\varepsilon]\right)\end{aligned}$$

Assuming that $\max_{1\le k\le n}\mathbb{P}[\|S_k\|_t>\varepsilon]<\frac{1}{3}$ implies that

$$\tfrac{1}{3}\mathbb{P}[M_n>3\varepsilon]\le\mathbb{P}[\|S_n\|_t>\varepsilon]$$

If $\max_{1\le k\le n}\mathbb{P}[\|S_k\|_t>\varepsilon]\ge\frac{1}{3}$ then the conclusion is trivial. □

Lemma 1.21. *Consider $\{S_n(t)\}$ as before. If*

$$\lim_{n,m\to\infty}\mathbb{P}[\|S_n-S_m\|_t>\varepsilon]=0\quad\text{for any }\varepsilon>0, \tag{6}$$

then there is a stochastic process $\{S(s):s\in[0,t]\}$ such that the sample functions of $S(s)$ belong to $D([0,\infty],\mathbb{R}^d)$ a.s. and

$$\lim_{n\to\infty}\|S_n-S\|_t=0\quad a.s. \tag{7}$$

Proof. By lemma 1.20

$$\mathbb{P}\left[\max_{n\le j\le m}\|S_j-S_n\|_t>3\varepsilon\right]\le 3\max_{n\le j\le m}\mathbb{P}[\|S_j-S_n\|_t>\varepsilon]$$

Triangle inequality implies that

$$\mathbb{P}\left[\max_{\substack{n\le j\le m\\ n\le k\le m}} \|S_j - S_k\|_t > 6\varepsilon\right] \le \mathbb{P}[\max_{n\le j\le m} \|S_j - S_n\|_t > 3\varepsilon]$$

Hence

$$\mathbb{P}\left[\max_{\substack{j\ge n\\ k\ge n}} \|S_j - S_k\|_t > 6\varepsilon\right] \le \mathbb{P}[\max_{j\ge n} \|S_j - S_n\|_t > 3\varepsilon]$$

Since the right–hand side converges to 0 as $n \to 0$, it follows that

$$\lim_{n\to\infty} \sup_{\substack{j\ge n\\ k\ge n}} \|S_j - S_k\|_t = 0 \quad \text{a.s}$$

As the function space $D([0,t],\mathbb{R}^d)$ is complete under the uniform norm, we get $\{S(s)\}$ with the desired properties. □

Lemma 1.22. *(Kolmogorov's Inequality) Let (X_n) is a sequence of independent random variables on $\mathbb{R}^d$ such that $\mathbb{E}\left[|X_n|^2\right] < \infty$ and $\mathbb{E}[X_n] = 0$ for each n. Let $S_n = \sum_{j=1}^n X_j$. Then*

$$\mathbb{P}\left[\sup_{n\ge 1} |S_n| > \varepsilon\right] \le \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty} \mathbb{E}[|X_n|^2] \tag{8}$$

for each $\varepsilon > 0$.

Proof. Given $1 \le n \le N$, note that

$$|S_N|^2 - |S_n|^2 = |S_N - S_n|^2 + 2(S_N - S_n)\cdot S_n \ge 2(S_N - S_n)\cdot S_n;$$

and therefore, since $S_N - S_n$ has mean–value 0 and is independent from $\sigma(X_1,\dots,X_n)$,

$$\mathbb{E}\left[|S_N|^2, A_n\right] \ge \mathbb{E}\left[|S_n|^2, A_n\right] \quad \text{for any } A_n \in \sigma(X_1,\dots,X_n). \tag{9}$$

In particular, if $A_1 = [|S_1| > \varepsilon]$ and

$$A_{n+1} = \left[\max_{1\le j\le n} |S_j| \le \varepsilon < |S_{n+1}|\right]$$

then, the A_n's are mutually disjoint and

$$B_N = \left[\max_{1\le n\le N} \|S_n\| > \varepsilon\right] = \bigcup_{n=1}^N A_n,$$

and so (9) implies that

$$\begin{aligned}\mathbb{E}[|S_N|^2, B_N] &= \sum_{n=1}^{N} \mathbb{E}[|S_N|^2, A_n] \geq \sum_{n=1}^{N} \mathbb{E}[|S_n|^2, A_n] \\ &\geq \varepsilon^2 \sum_{n=1}^{N} \mathbb{P}[A_n] = \varepsilon^2 \mathbb{P}[B_N].\end{aligned}$$

This leads to

$$\begin{aligned}\varepsilon^2 \mathbb{P}\left[\sup_{n\geq 1} |S_n| > \varepsilon\right] &= \lim_{N\to\infty} \varepsilon^2 \mathbb{P}[B_N] \\ &\leq \lim_{N\to\infty} \mathbb{E}\left[|S_N|^2\right] \leq \sum_{n=1}^{\infty} \mathbb{E}[|X_n|^2].\end{aligned}$$

This finishes up the proof. □

1.2. **Poisson Processes.**

Definition 1.13. A stochastic process $\{X_t : t \geq 0\}$ on $\mathbb{R}$ is a Poisson process with parameter $c > 0$ if it is a Lévy process and for any $t > 0$, X_t has a Poisson distribution with mean ct

Definition 1.14. Let $\{Z_n : n = 1, \ldots\}$ be a sequence of i.i.d random variables on $\mathbb{R}^d$. Let $S_0 = 0$, and $S_n = \sum_{j=1}^{n} Z_j$ for $n = 1, 2, \ldots$. Then the random sequence $\{S_n : n = 0, 1, \ldots\}$ is called random walk on $\mathbb{R}^d$.

Theorem 1.23. *(Construction) Let $\{W_n : n = 0, 1, \ldots\}$ be a random walk on $\mathbb{R}$ such that $T_n = W_n - W_{n-1}$ has exponential distribution with mean $c > 0$. Define X_t by*

$$X_t(\omega) = n \quad iff \quad W_n(\omega) \leq t < W_{n+1}(\omega)$$

Then $\{X_t : t \geq 0\}$ is a Poisson process.

Proof. Since T_n is nonnegative for any n, it follows that $W_n \leq W_{n+1}$. From the estimate

$$\begin{aligned}\mathbb{P}\left[\lim_{n\to\infty} W_n \leq t\right] &= \lim_{n\to\infty} \mathbb{P}\left[W_n \leq t\right] \\ &\leq \lim_{n\to\infty} \mathbb{P}\left[T_1 \leq t, T_2 \leq t, \ldots, T_n \leq t\right] \\ &= \lim_{n\to\infty} \left(\mathbb{P}\left[T_1 \leq t\right]\right)^n = 0\end{aligned}$$

we conclude that W_n increases to ∞ a.s., that the sample functions of X_t are nondecreasing right continuous step functions with left limits and jumps of size 1, and that

$$X_t = \max\{n : W_n \leq t\} < \infty \qquad \text{a.s}$$

The estimate

$$\mathbb{P}[X_0 = 0] = \mathbb{P}[W_1 > 0] = \mathbb{P}[T_1 > 0] = 1$$

implies that $X_0 = 0$ a.s.

Being W_n the sum of n exponential distributed distributions with parameter c, it follows that W_n has $\Gamma(c, n)$–distribution. Thus,

$$\mathbb{P}[X_t = n] = \mathbb{P}[W_n \le t < W_n + T_{n+1}] = \mathbb{P}[(W_n, T_{n+1}) \in B]$$

where $B = \{(x, y) : 0 \le x \le t < x + y\}$. Hence

$$\mathbb{P}[X_t = n] = \tfrac{c^{n+1}}{(n-1)!} \int_0^t \int_{t-x}^{\infty} x^{n-1} e^{-cx} e^{-cy}\, dy\, dx = \tfrac{e^{-ct}}{n!}(ct)^n \tag{10}$$

This shows that X_t is Poisson–ct distributed. Next, we show that

$$\mathbb{P}[W_{n+1} > t + s | X_t = n] = e^{-cs} = \mathbb{P}[T_1 > s] \tag{11}$$

A computation similar to the one used to obtain equation (10) leads to

$$\begin{aligned} \mathbb{P}[X_t &= n, W_{n+1} > t + s] = \mathbb{P}[W_n \le t, W_n + T_{n+1} > t + s] \\ &= \tfrac{c^{n+1}}{(n-1)!} \int_0^t x^{n-1} e^{-cx} \int_{t+s-x}^{\infty} e^{-cy}\, dy\, dx \\ &= \tfrac{e^{-c(t+s)}}{n!}(ct)^n \end{aligned}$$

This and (10) yield (11).

Let $n \ge 0$ and $m \ge 1$. Consider the conditional distribution of $(W_{n+1} - t, T_{n+2}, \dots, T_{n+m})$ given that $X_t = n$. We will show below that it is equal to the distribution of $(T_1, \dots, T_m)$. Let us set $\mathbb{P}[X_t = n] = a$, and observe that, for any $s_1, \dots, s_m \ge 0$,

$$\begin{aligned} \mathbb{P}[W_{n+1} &- t > s_1, T_{n+2} > s_2, \dots, T_{n+m} > s_m | X_t = n] \\ &= \mathbb{P}[W_n \le t, W_{n+1} - t > s_1, T_{n+2} > s_2, \dots, T_{n+m} > s_m]/a \\ &= \mathbb{P}[W_n \le t, W_{n+1} - t > s_1]\mathbb{P}[T_{n+2} > s_2, \dots, T_{n+m} > s_m]/a \\ &= \mathbb{P}[W_{n+1} - t > s_1 | X_t = n]\mathbb{P}[T_{n+2} > s_2, \dots, T_{n+m} > s_m] \\ &= \mathbb{P}[T_1 > s_1]\mathbb{P}[T_2 > s_2, \dots, T_m > s_m] \\ &= \mathbb{P}[T_1 > s_1, \dots, T_m > s_m] \end{aligned}$$

The argument above will show that for $t > 0$ and $s > 0$

$$\mathbb{P}[X_{t+s} - X_t = m] = \mathbb{P}[X_s = m] \tag{12}$$

for

$$\begin{aligned} \mathbb{P}[X_t = n, X_{t+s} - X_t = m] &= \mathbb{P}[X_t = n]\mathbb{P}[X_{t+s} = m + n | X_t = n] \\ &= \mathbb{P}[X_t = n]\mathbb{P}[W_{m+n} \le t + s < W_{m+n+1} | X_t = n] \end{aligned}$$

and since $W_{m+n} \leq t+s < W_{m+n+1}$ can be written as

$$W_{n+1} - t + T_{n+2} + \ldots + T_{m+n} \leq s < W_{n+1} - t + T_{n+2} + \ldots + T_{m+n+1}$$

it follows that

$$\begin{aligned}\mathbb{P}[X_t = n, X_{t+s} - X_t = m] &= \mathbb{P}[X_t = n]\mathbb{P}[W_m \leq s < W_{m+1}] \\ &= \mathbb{P}[X_t = n]\mathbb{P}[X_s = m]\end{aligned}$$

Addition over n yields (12).
The same argument shows that for, $0 \leq t_0 < t_1 < \ldots < t_k$,

$$\begin{aligned}&\mathbb{P}[X_{t_0} = n_0, X_{t_1} - X_{t_0} = n_1, \ldots, X_{t_k} - X_{t_{k-1}} = n_k] \\ &\quad = \mathbb{P}[X_{t_0} = n_0, X_{t_1} = n_0 + n_1, \ldots, X_{t_k} = n_0 + \ldots + n_k] \\ &\quad = \mathbb{P}[X_{t_0} = n_0]\mathbb{P}[X_{t_1 - t_0} = n_1, \ldots, X_{t_k - t_0} = n_1 + \ldots + n_k]\end{aligned}$$

Respiting this, we get the independent increments property, as

$$\begin{aligned}&\mathbb{P}[X_{t_0} = n_0, X_{t_1} - X_{t_0} = n_1, \ldots, X_{t_k} - X_{t_{k-1}} = n_k] \\ &\quad = \mathbb{P}[X_{t_0} = n_0]\mathbb{P}[X_{t_1 - t_0} = n_1] \cdots \mathbb{P}[X_{t_k - t_{k-1}} = n_k] \\ &\quad = \mathbb{P}[X_{t_0} = n_0]\mathbb{P}[X_{t_1} - X_{t_0} = n_1] \cdots \mathbb{P}[X_{t_k} - X_{t_{k-1}} = n_k]\end{aligned}$$

To see that the process is continuous in probability, notice that if X_t is discontinuous in probability at time t_0, then since X_t is nondecreasing, there is a set A of probability larger than 0 where $X_{t_0} - X_{t_0-} \geq 1$. But $A \subset \cup_{n=1}^{\infty}[W_n = t_0]$, which has probability 0. $\square$

Using Poisson processes we can construct another example of a Lévy process.

Definition 1.15. A distribution μ on $\mathbb{R}^d$ is compound Poisson if for some $c > 0$, and for some distribution σ on $\mathbb{R}^d$ with $\sigma(\{0\}) = 0$

$$\widehat{\mu}(z) = \exp\left(c(\widehat{\sigma}(z) - 1)\right) \qquad z \in \mathbb{R}^d$$

The Poisson distribution is a special case where $d = 1$ and $\sigma = \delta_1$.

Definition 1.16. Let $c > 0$ and let σ be a distribution with $\sigma(\{0\}) = 0$. A stochastic process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ is a compound Poisson process associated with c and σ if it is a Lévy process, and for $t > 0$, X_t has a compound Poisson distribution

$$\mathbb{E}\left[e^{i\langle z, X(t)\rangle}\right] = \exp\left(ct(\widehat{\sigma}(z) - 1)\right) \qquad z \in \mathbb{R}^d$$

As we will see, the $c > 0$ and the distribution σ are determined uniquely by $\{X_t\}$.

Theorem 1.24. *(Construction)*
Let $\{N_t : t \geq 0\}$ be a Poisson process and $\{S_n : n = 0 \dots\}$ be a random walk on $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that $\{N_t\}$ and $\{S_n\}$ are independent and that $\mathbb{P}[S_1 = 0] = 0$. Define

$$X_t(\omega) = S_{N_t(\omega)}(\omega)$$

Then $\{X_t : t \geq 0\}$ is a compound Poisson process with σ as the distribution of S_1.

Proof. Let $B, B_0, B_1, \dots, \in \mathcal{B}(\mathbb{R}^d)$. Then,

$$\begin{aligned}\mathbb{P}[X_t] = \mathbb{P}\left[S_{N_t} \in B\right] &= \sum_{n=0}^{\infty} \mathbb{P}\left[N_t = n, S_n \in B\right] \\ &= \sum_{n=0}^{\infty} \mathbb{P}[N_t = n]\mathbb{P}[S_n \in B]\end{aligned}$$

For $0 \leq t_0 < t_1$, the stationary and independent increments property of the Poisson process and of the random walk lead to

$$\begin{aligned}&\mathbb{P}[X_{t_0} \in B_0, X_{t_1} - X_{t_0} \in B_1] = \mathbb{P}\left[S_{N_{t_0}} \in B_0, S_{N_{t_1}} - S_{N_{t_0}} \in B_1\right] \\ &\quad = \sum_{n_0, n_1} \mathbb{P}[N_{t_0} = n_0, N_{t_1} - N_{t_0} = n_1, S_{n_0} \in B_0, S_{n_1 + n_0} - S_{n_0} \in B_1] \\ &\quad = \sum_{n_0, n_1} \mathbb{P}[N_{t_0} = n_0]\mathbb{P}[N_{t_1 - t_0} = n_1]\mathbb{P}[S_{n_0} \in B_0]\mathbb{P}[S_{n_1} \in B_1] \\ &\quad = \mathbb{P}[X_{t_0} \in B_0]\mathbb{P}[X_{t_1} \in B_1]\end{aligned}$$

Taking $B_0 = \mathbb{R}^d$ we get that $X_{t_1} - X_{t_0} \stackrel{d}{=} X_{t_1 - t_0}$. In general, for $0 \leq t_0 < \dots < t_k$,

$$\begin{aligned}&\mathbb{P}[X_{t_0} \in B_0, X_{t_1} - X_{t_0} \in B_1, \dots, X_{t_k} - X_{t_{k-1}} \in B_k] \\ &\quad = \sum_{n_0, \dots, n_k} \mathbb{P}[N_{t_0} = n_0, N_{t_1} - N_{t_0} = n_1, \dots, N_{t_k} - N_{t_{k-1}} = n_k, S_{n_0} \in B_0, \\ &\qquad\qquad S_{n_0 + n_1} - S_{n_0} \in B_1, \dots, S_{n_0 + \dots + n_k} - S_{n_0 + \dots + n_{k-1}} \in B_k] \\ &\quad = \sum \mathbb{P}[N_{t_0} = n_0]\mathbb{P}[N_{t_1 - t_0} = n_1] \cdots \mathbb{P}[N_{t_k} - N_{t_{k-1}} = n_k] \\ &\qquad \times \mathbb{P}[S_{n_0} \in B_0]\mathbb{P}[S_{n_1} \in B_1] \cdots \mathbb{P}[S_{n_k} \in B_k] \\ &\quad = \mathbb{P}[X_{t_0} \in B_0]\mathbb{P}[X_{t_1} - X_{t_0} \in B_1] \cdots \mathbb{P}[X_{t_k} - X_{t_{k-1}} \in B_k]\end{aligned}$$

The right continuity with left limits condition for sample paths of $\{X_t\}$, as well as the continuity in probability, follow from $\{N_t\}$ fulfilling the same conditions. By the same token, $X_0 = 0$ a.s. As to the characteristic function,

$$\begin{aligned}\mathbb{E}\left[e^{i\langle z,X(t)\rangle}\right] &= \sum_{n=0}^{\infty}\mathbb{P}[N_t = n]\mathbb{P}\left[e^{i\langle z,S(n)\rangle}\right]\\ &= \sum_{n=0}^{\infty}\tfrac{e^{-ct}}{n!}(ct)^n\widehat{\sigma}(z)^n = \exp[ct(\widehat{\sigma}(z)-1)]\end{aligned}$$

as asserted □

1.3. **Brownian motion.** Another important example of a Lévy process is Brownian motion, very often referred as Wiener Process.

Definition 1.17. A stochastic process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a Brownian motion, or a Wiener process, if it is a Lévy process, and if for any $t > 0$, X_t has a Gaussian distribution with mean 0 and covariance matrix tI.

There are many ways to prove the existence of Brownian motion. Later on, we will derive a general and standard method to construct Lévy processes. We borrow from [1] the following construction of Brownian motion for $d = 1$.

Theorem 1.25. *(Existence and Continuity of Brownian motion)*

(i) There exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a family of random variables $\{W_t : t \geq 0\}$ on it that has stationary independent increments, and such that $W_0 = 0$ and the law of the increment $W_t - W_s$ is $N(0, t-s)$.

(ii) Given such a family, one can change W_t for each t on a negligible set in such a way that for every $\omega \in \Omega$ the path $t \mapsto W_t(\omega)$ is a continuous function.

Proof. Let $(\Omega, \mathcal{F}, \mathbb{P})$ any probability space that admits a sequence (ξ_n) of independent random variables each having law $N(0,1)$. For instance, take countably many copies of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma_1)$[1] and let $(\Omega, \mathcal{F}, \mathbb{P})$ be their product. The projections ξ_n over the nth–component will do the job. Let (ϕ_n) be any orthonormal basis of $L^2(\mathbb{R})$; thus, any $f \in L^2(\mathbb{R})$ can be expressed uniquely in the form

$$f = \sum_{n=1}^{\infty} a_n\phi_n, \quad \text{with} \quad \|f\|_{L^2([0,\infty))} = \sum_{n=1}^{\infty} a_n^2.$$

[1] $\gamma_1(dx) = (1/\sqrt{2\pi})e^{-x^2/2}\,dx$

We define the map $\Phi : L^2([0,\infty)) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\Phi(f) = \sum_{n=1}^{\infty} a_n \xi_n.$$

Notice that Φ associates to each equivalence class in $L^2([0,\infty))$ an equivalence class in $L^2(\Omega)$, also it is clear that Φ is a linear isometry from $L^2([0,\infty))$ into $L^2(\Omega)$.

The law of any function in the equivalence class $\Phi(f)$ is $N(0, \|f\|_{L^2(\mathbb{R})})$. To show this, note that if $f = \sum_n a_n \phi_n$ a.e. then

$$\int_0^{\infty} f^2(t)\, dt = \sum_n a_n^2 = \mathbb{E}\left[\Phi(f)^2\right]$$

The simple calculation

$$\mathbb{E}\left[e^{i\,z\Phi(f)}\right] = \mathbb{E}\left[e^{i\,z\sum_n a_n \xi_n}\right] = \prod_n \mathbb{E}\left[e^{i\,z a_n \xi_n}\right] = e^{-z^2 \sum_n a_n^2/2}$$

leads to the desired result.

A similar argument shows that if $f_1, f_2, \ldots$ are orthogonal then not only are $\Phi(f_1), \Phi(f_2), \ldots$ orthogonal in $L^2(\Omega)$, but actually they are independent:

$$\left\| \sum_k \alpha_k f_k \right\|^2_{L^2([0,\infty))} = \sum_k \alpha_k^2 \|f_k\|^2_{L^2([0,\infty))}$$

therefore

$$\begin{aligned}
\mathbb{E}\left[e^{i\,\sum_k \alpha_k \Phi(f_k)}\right] &= \mathbb{E}\left[e^{i\,\Phi\left(\sum_k \alpha_k f_k\right)}\right] = e^{-\|\sum_k \alpha_k f_k\|^2/2} \\
&= \prod_k e^{-\alpha_k^2 \|f_k\|^2/2} = \prod_k \mathbb{E}\left[e^{i\,\alpha_k \Phi(f_k)}\right]
\end{aligned}$$

For any $t \geq 0$, let $\dot{W}_t$ be the class of equivalence $\Phi\left(\mathbf{1}_{[0,t]}\right)$ and now we simply choose $W_t \in \dot{W}_t$. If $0 \leq s < t$ then $\dot{W}_t - \dot{W}_s = \Phi\left(\mathbf{1}_{[s,t]}\right)$ is $N(0, t-s)$ distributed, thus stationarity increments property follows. Since disjoint intervals are $L^2([0,\infty))$–orthogonal, the independence of increments follows immediately. This is part (i) of the theorem.
The second part is much more difficult to prove. It is based on the following result, due to Kolmogorov:

Proposition 1.26. *Let U be an open subset of $\mathbb{R}^d$, and let $\left\{\widehat{X}_u : u \in U\right\}$ be a collection of equivalence classes of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in some complete metric* *space (E,ρ). Assume that there exist non–negative constants p,α and C such that*

$$\mathbb{E}\left[\rho\left(\widehat{X}_u,\widehat{X}_v\right)^p\right]\leq C|u-v|^{d+\beta}\quad \textit{for}\, u,v\in U.$$

Then,

(i) *There exists a selection $\left\{X_u\in\widehat{X}_u : u\in U\right\}$ such that for every $\omega\in\Omega$, the map $u\mapsto X_u(\omega)$ from U to E is continuous.*
(ii) *Moreover, for any $\lambda\in(0,\beta/p)$ and for any $\alpha>0$, there is $\Omega_{\alpha,\lambda}\in\mathcal{F}$ with $\mathbb{P}[\Omega_{\alpha,\lambda}]>1-\alpha$ such that the family*

$$\{u\mapsto X_u(\omega):\omega\in\Omega_{\alpha,\lambda}\}$$

of E–valued functions can be chosen to be uniformly Hölder continuous in any compact subset K of U with exponent λ.
(iii) *Any two continuous selections $X_u, X'_u\in\widehat{X}_u$ are indistinguishable in the sense that*

$$\mathbb{P}\left[\sup_{u\in U}|X_u-X'_u|>0\right]=0$$

A proof of this powerful result can be found in [1] and [6].

Continuation of proof of 1.25: In order to have a parameter domain we extend the family $\{\widehat{W_t}\}$ constructed in part (i) to negative times. One possibility is to set $\widehat{W_t}=\widehat{W_{-t}}$ for $t>0$. Then the estimate

$$\mathbb{E}\left[|\widehat{W_t}-\widehat{W_s}|^c\right]\leq|t-s|^{c/2}\mathbb{E}\left[|\widehat{W_1}|^c\right]$$

gives the desired result. □

Incidentally, we have shown that almost surely, the maps W_t are locally Hölder with exponent λ for all $0<\lambda<1/2$. It is shown in [4] that $1/2$ is a threshold value, which consequently shows that almost surely all paths of Brownian motion are nowhere differentiable. We can, however, get aground the previous fact and prove by a direct method the nowhere differentiability of Brownian motion.

Theorem 1.27. *(Wiener) Let be W standard Brownian motion in some probability space $(\Omega,\mathcal{F},\mathbb{P})$. Then except from a set H of measure 0, the paths $t\mapsto W_t(\omega)$ are nowhere differentiable.*

Proof. The idea is to check that the set $H\subset\Omega$ where the path $t\mapsto W_t(\omega)$ is differentiable at some point s is contained in

$$B=\bigcup_{K=1}^{\infty}\bigcup_{M=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\bigcup_{k=1}^{Kn}\bigcap_{j=k}^{k+2}\left[\left|W_{(j+1)/n}-W_{j/n}\right|\leq\tfrac{7M}{n}\right].$$

To this end, suppose $t \mapsto W_t(\omega)$ is differentiable at some point s. Then for some $K \in \mathbb{N}$, $s < K$. Also, there exist $M, N \in \mathbb{N}$ such that for all $t \in (s - 5/n, s + 5/n)$, $|W_t(\omega) - W_s(\omega)| \le M \cdot |t - s|$. Consider the first consecutive number of the form j/n with $j \in \mathbb{N}$ in $(s, s + 5/n)$. The triangle inequality gives

$$\left|W_{(j+1)/n} - W_{j/n}\right| \le \left|W_{(j+1)/n} - W_s\right| + \left|W_s - W_{j/n}\right|$$

and our assertion follows. To prove that B is indeed negligible, we show that

$$\begin{aligned} Q &\stackrel{\text{def}}{=} \mathbb{P}\left[\bigcap_{n=N}^{\infty} \bigcup_{k=1}^{Kn} \bigcap_{j=k}^{k+2} \left[\left|W_{(j+1)/n} - W_{j/n}\right| \le \tfrac{7M}{n}\right]\right] \\ &\le \liminf_{n\to\infty} \mathbb{P}\left[\bigcup_{k=1}^{Kn} \bigcap_{j=k}^{k+2} \left[\left|W_{(j+1)/n} - W_{j/n}\right| \le \tfrac{7M}{n}\right]\right] = 0 \end{aligned}$$

To verify this, note that the events

$$\left[\left|W_{(j+1)/n} - W_{j/n}\right| \le \tfrac{7M}{n}\right] \qquad j = k, k+1, k+2$$

are independent and, by the homogeneity property, each has probability

$$\begin{aligned} \mathbb{P}\left[\left|W_{1/n}\right| \le 7M/n\right] &= \tfrac{1}{\sqrt{2\pi/n}} \int_{-7M/n}^{7M/n} e^{-x^2 n/2}\, dx \\ &= \tfrac{1}{\sqrt{2\pi}} \int_{-7M/\sqrt{n}}^{7M/\sqrt{n}} e^{-y^2/2}\, dy \le \tfrac{14M}{\sqrt{2\pi n}} \end{aligned}$$

Hence

$$Q \le \liminf_{n} Kn \left(\tfrac{const}{\sqrt{n}}\right)^3 = 0$$

concluding thus the proof. $\square$

An immediate consequence of the las theorem is

Corollary 1.28. *(d=1) Almost surely there is no interval in which $t \mapsto W_t(\omega)$ is monotone.*

Now that we have shown the existence of Brownian motion for $d = 1$, we can construct Brownian motion in higher dimensions.

Proposition 1.29. *Let be $\{W_t : t \ge 0\}$ be a stochastic process on $\mathbb{R}^d$ and let $W_1(t), \dots, W_d(t)$ be the components of W_t. The the following are equivalent.*

- *(i) $\{W_t\}$ is a d–dimensional Brownian motion*
- *(ii) $\{W_j(t)\}$ is a one–dimensional Brownian motion for each j and $\{W_1(t)\}, \dots, \{W_d(t)\}$ are independent.*

See [7] for the details of the proof.
The following theorem reveals some invariance properties of Brownian motion under some transformations.

Theorem 1.30. *Let $\{W_t\}$ be a d–dimensional Brownian motion*

(i) $\{-W_t\}$ *is a d–dimensional Brownian motion.*
(ii) *For each $c > 0$, $\{c^{-1/2}W_{ct}\}$ is a d–dimensional Brownian motion*
(iii) *Define $X_t = tW_{t^{-1}}$ for $t > 0$ and $X_0 = 0$. Then $\{X_t : t \geq 0\}$ is a d–dimensional Brownian motion.*

Proof. Only (iii) requires some extra thought. It is clear that X_t Gaussian distributed with mean 0 and variance T. For any $s, t > 0$ the estimate

$$\mathbb{E}\left[X_j(t)X_k(t)\right] = st\mathbb{E}\left[W_j(s^{-1})W_k(t^{-1})\right] = \delta_{jk}(s \wedge t)$$

tells that $X_t \overset{d}{=} W_t$ for $t > 0$. Consider the continuous version W_t of standard Brownian motion. Then X_t is continuous on $(0, \infty)$. We need to show that $X_t \to 0$ as $t \to 0$ almost surely. To this end consider the set

$$\Omega_1 = \bigcap_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcap_{t\in\mathbb{Q}\cap(0,1/m)}[|W_t| \leq 1/n]$$

by the same token, we define Ω_1' with X_t in place of W_t. By our choice of W_t,

$$\Omega_1 = \left[\lim_{t\downarrow 0} W_t = 0\right] = [W_0 = 0] = \Omega$$

The identity in law of X_t and W_t for $t > 0$ gives $\mathbb{P}[\Omega_1'] = \mathbb{P}[\Omega] = 1$, concluding thus our proof. □

Theorem 1.31. *(Behavior for large t). ($d = 1$) For any sequence $t_n \uparrow \infty$*

$$\limsup_{n\to\infty} W_{t_n} = \infty \quad a.s. \tag{13}$$

$$\liminf_{n\to\infty} W_{t_n} = -\infty \quad a.s. \tag{14}$$

Proof. First notice that $W_{t_n} \overset{d}{=} t_n^{1/2}W_1$, thus for any $a > 0$

$$\mathbb{P}[W_{t_n} > a] = \mathbb{P}[W_1 > t_n^{-1/2}a] \to 1/2, \qquad n \to 0$$

Then an application of Fatou's lemma leads to

$$\begin{aligned}\mathbb{P}\left[\limsup_n [W_{t_n} > a]\right] &= \mathbb{P}[\limsup_n W_{t_n} > a] \\ &\geq \limsup_n \mathbb{P}[W_{t_n} > a] = 1/2\end{aligned}$$

Therefore

$$\mathbb{P}[\limsup_n W_{t_n} = \infty] \leq 1/2$$

Let $t_0 = 0$ and $Z_n = W_{t_n} - W_{t_{n-1}}$. Then $\{Z_n\}$ are independent and

$$W_{t_n} = \sum_{k=1}^{n} Z_n$$

We have that

$$\begin{aligned} A = \left[\limsup_n W_{t_n} = \infty\right] &= \left[\limsup_n (W_{t_n} - W_{t_m}) = \infty\right] \\ &\in \sigma(Z_{m+1}, Z_{m+2}, \ldots) \end{aligned}$$

for each m. Thus the event on the left hand side above is a tail event. Then by Kolmogorov's 0–1 law, we have that $\mathbb{P}[A] = 1$. The symmetry implied by theorem 1.30(i) gives (14) from (13). □

Theorem 1.32. *(Behavior for small t)($d = 1$) Let*

$$\begin{aligned} T_0(\omega) &= \inf\{t > 0 : W_t(\omega) > 0\} \\ T_0'(\omega) &= \inf\{t > 0 : W_t(\omega) < 0\} \end{aligned}$$

for $\omega \in \Omega$. Then

$$T_0 = 0 \qquad a.s. \tag{15}$$

$$T_0' = 0 \qquad a.s. \tag{16}$$

Proof. Let $t_n \downarrow 0$. We use $X_t = tW_{t^{-1}}$. Theorems 1.30 and 1.31 give

$$\begin{aligned} \mathbb{P}\left[\limsup_n [W_{t_n} > 0]\right] &= \mathbb{P}\left[\limsup_n [X_{t_n} > 0]\right] \\ &= \mathbb{P}\left[\limsup_n [W_{t_n^{-1}} > 0]\right] = 1 \end{aligned}$$

This proves (15). Using symmetry we get (16) from (15). □

Now, we give another characterization of Brownian motion based on theorem 1.19.

Theorem 1.33. *(Lévy) Let X be a continuous process in $\mathbb{R}^d$ with independent increments. Then $X - X_0$ is Gaussian, and there are continuous functions b in $\mathbb{R}^d$ and a in $\mathbb{R}^{d^2}$, the latter with nonnegative definitive increments such that $X_t - X_s$ is $N(b_t - b_s, c_t - c_s)$ for any $s \leq t$.*

Proof. We fix $s < t$ and consider successive divisions of the interval $[s,t]$ in n subintervals of equal length. Let us denote the corresponding increments by $\xi_{n1},\dots,\xi_{nn}$. By continuity we have that $\max_j|\xi_{nj}|\to 0$ a.s, thus by theorem (1.19) $X_t - X_s = \sum_j \xi_{nj}$ is a Gaussian variable. Since X has independent increments, it follows that the process $X - X_0$ is Gaussian. Define $b-t = \mathbb{E}[X_t - X_0]$ and $a_t = \mathrm{cov}(X_t - X_0)$. Then $\mathbb{E}[X_t - X_s] = b_t - b_s$ and by independence

$$0 \le \mathrm{cov}(X_t - X_s) = \mathrm{cov}(X_t) - \mathrm{cov}(X_s) = a_t - a_s, \quad \text{for } s < t$$

The continuity of X implies that $X_s \xrightarrow{d} X_t$ as $s \to t$ and thus $b_s \to b_t$ and $a_s \to a_t$. Proving continuity of b_t and c_t. □

2. Lévy processes: Existence and characterization

2.1. **Infinitely divisible laws and Lévy processes in law.** The basic results on this section are the existence of Lévy processes in law, the Lévy–Kintchine theorem, which gives a characterization of finitely divisible distributions in terms of their characteristic functions. We will give an analytic proof for this theorem and then we will use it later to prove the Ito–Lévy representation theorem of additive processes, which roughly speaking, decomposes such processes in a continuous part plus a pure jump part. In the last section, it will show that we can go the other way around and start from the Ito–Lévy's theorem and derive from it the Lévy–Kintchine theorem. There we will furnish a probabilistic proof.

Definition 2.1. A probability measure μ on $\mathbb{R}^d$ is *infinitely divisible* if for any $n \in \mathbb{N}$, there is a probability measure μ_n on $\mathbb{R}^d$ such that $\mu = \mu_n^n$ where

$$\mu_n^n = \underbrace{\mu_n * \ldots * \mu_n}_{n}$$

Simple examples of such distributions are Gaussian, exponential, compound Poisson, delta and Cauchy distributions.

It is easy to check that if $\{X_t\}$ is a Lévy process on $\mathbb{R}^d$, then for every t, the distribution of X_t is infinitely divisible. The following lemma follows immediately from the definition

Lemma 2.1. *If μ_1 and μ_2 are infinitely divisible, then $\mu_1 * \mu_2$ is infinitely divisible.*

The following results will give the necessary means to show the existence of Lévy processes in law.

Lemma 2.2. *If μ is infinitely divisible, then $\widehat{\mu}$ has no zeroes, that is $\widehat{\mu}(z) \neq 0$ for any $z \in \mathbb{R}^d$.*

Proof. For each n there is a law μ_n such that $\widehat{\mu}(z) = \widehat{\mu}_n^n(z)$. By theorem 1.16 (v), $|\widehat{\mu}_n(z)|^2 = |\widehat{\mu}(z)|^{2/n}$ is a characteristic function. Let

$$\varphi(z) = \lim_{n\to\infty} |\widehat{\mu}(z)|^{2/n} = \lim_{n\to\infty} |\widehat{\mu}_n(z)|^2 = \begin{cases} 1 & \text{if } \widehat{\mu}(z) \neq 0 \\ 0 & \text{if } \widehat{\mu}(z) = 0 \end{cases}$$

Since $\widehat{\mu}(0) = 1$ and $\widehat{\mu}$ is continuous, then $\varphi(z) = 1$ in a neighborhood of 0. It follows from 1.16 (viii), that $\varphi(z)$ is a characteristic function for some law, and thus continuous. Hence $\varphi(z) \equiv 1$ which implies that $\mu(z) \neq 0$ for any $z \in \mathbb{R}^d$. □

Next we borrow three basic results from Complex Variable. The proofs can be found in [7] or in [5] as exercises.

Lemma 2.3. *For any $u \in \mathbb{R}$ and $n \in \mathbb{N}$*

$$e^{i\,u} = \sum_{k=0}^{n-1} \tfrac{(i\,u)^k}{k!} + \theta(u)\tfrac{|u|^n}{n!}$$

with $|\theta(u)| \in \mathbb{C}$ satisfying $\theta(u) \leq 1$

Lemma 2.4. *(Continuous logarithm) Suppose that $\varphi : \mathbb{R}^d \mapsto \mathbb{C}$ is a continuous function such that $\varphi(0) = 1$ and $\varphi(z) \neq 0$ for any $z \in \mathbb{R}^d$. Then, there is a unique function $f : \mathbb{R}^d \mapsto \mathbb{C}$ such that $f(0) = 0$ and $e^{f(z)} = \varphi(z)$. For any $n \in \mathbb{N}$ there is a unique continuous function $g_n : \mathbb{R}^d \mapsto \mathbb{C}$ such that $g_n(0) = 1$ and $g_n^n(z) = \varphi(z)$. Functions f and g_n are related by*

$$g_n(z) = e^{f(z)/n}$$

.

We shall denote f by $\log\varphi$ and g_n as $\varphi^{1/n}$, but we must not interpret this as the composition of an analytic branch of logarithm with φ. More generally,

$$\varphi^t(z) = e^{t\,f(z)}$$

Lemma 2.5. *Suppose that $\varphi(z)$ and $\varphi_n(z)$, $n = 1, 2, \ldots$, are continuous functions from $\mathbb{R}^d$into $\mathbb{C}$ such that $\varphi(0) = 1 = \varphi_n(0)$, $\varphi(z) \neq 0$ and $\varphi_n(z) \neq 0$ for any $z \in \mathbb{R}^d$. If $\varphi_n \to \varphi$ uniformly on compact sets, then $\log\varphi_n \to \log\varphi$ uniformly on compact sets.*

Limits in law of infinitely divisible distributions are also infinitely divisible.

Lemma 2.6. *If (μ_k) is a sequence of infinitely divisible distributions and $\mu_k \xrightarrow{d} \mu$, then μ is infinitely divisible*

Proof. First we show that $\widehat{\mu}(z) \neq 0$ for any $z \in\mathbb{R}^d$. This will guaranty the existence of a continuous logarithm. By theorem 1.16 (vi), $\widehat{\mu}_k(z) \to \widehat{\mu}(z)$ uniformly in compact sets, hence $|\widehat{\mu_k}(z)|^{2/n} \to |\widehat{\mu}(z)|^{2/n}$ uniformly in compact sets for $n = 1, 2, \ldots$ as $k \to \infty$. Since each μ_k is infinitely divisible, $|\widehat{\mu}_k(z)|^{2/n}$ is a characteristic function; hence, $|\widehat{\mu}(z)|^{2/n}$ is continuous and then again, by Lévy continuity, it is a characteristic function as well. The simple identity $|\widehat{\mu}(z)|^2 = \big(|\widehat{\mu}(z)|^{2/n}\big)^n$ shows that $|\widehat{\mu}(z)|^2$ is the characteristic function of an infinitely divisible distribution. Therefore $\widehat{\mu}(z) \neq 0$ for any $z \in\mathbb{R}^d$.
By lemma 2.5, $\log \widehat{\mu_k}(z) \to \log \widehat{\mu}(z)$ uniformly on compact sets. Hence $\widehat{\mu_k}^{1/n}(z) \to \widehat{\mu}^{1/n}(z)$ uniformly in compact sets for any $n = 1, 2, \ldots$, as $k \to \infty$. Being $\widehat{\mu_k}^{1/n}(z)$ characteristic functions, and $\widehat{\mu}^{1/n}(z)$ continuous, it follows that $\widehat{\mu}^{1/n}(z)$ is a characteristic function for any $n \in \mathbb{N}$. Therefore, μ is an infinitely divisible distribution, since $\widehat{\mu}(z) = \big(\widehat{\mu}^{1/n}(z)\big)^n$.

Proposition 2.7. *If μ is infinitely divisible, then for every $t \in [0, \infty)$ μ^t is a definable infinitely divisible distribution.*

Proof. Since $\widehat{\mu}^{1/n}(z) = \big(\widehat{\mu}^{1/nk}(z)\big)^k$, it follows that for any $n \in \mathbb{N}$ $\mu^{1/n}$ is an infinitely divisible distribution. Hence for any positive integers n and m, $\mu^{n/m}$ is also an infinitely divisible distribution. For any irrational number $t > 0$, choose a sequence (r_n) in $\mathbb{Q}$ such that $r_n \to t$. Then $\widehat{\mu}^{r_n}(z) \to \widehat{\mu}^t(z)$. The later is continuous, thus Lévy continuity shows that $\widehat{\mu}^t(z)$ is a characteristic function. Lemma 2.6 leads to positive conclusion. $\square$.

Using infinitely divisible distributions is all we need to construct Lévy processes in law.

Theorem 2.8. *(Existence of Lévy processes in law)*

(i) If $\{X_t : t \geq 0\}$ is a Lévy process in law on $\mathbb{R}^d$, then for any $t \geq 0$, $\mathbb{P}_{X_t}$ is infinitely divisible, and letting $\mathbb{P}_{X_1} = \mu$, we have that $\mathbb{P}_{X_t} = \mu^t$.
(ii) Conversely, if μ is an infinitely divisible distribution on $\mathbb{R}^d$, then there is a Lévy process in law $\{X_t : t \geq 0\}$ such that $\mathbb{P}_{X_1} = \mu$.
(iii) If $\{X_t\}$ and $\{X'_t\}$ are Lévy processes in law on $\mathbb{R}^d$ such that $\mathbb{P}_{X_1} = \mathbb{P}_{X'_1}$, then $X \overset{d}{=} X'$.

Proof. (i) Let $\{X_t\}$ be a Lévy processes in law. For any $t > 0$ consider the points $t_k = kt/n$ with $k = 0, \ldots, n$, with $n \in \mathbb{Z}_+$. Since

$$X_t = \sum_{k=1}^{n} \big(X_{t_k} - X_{t_{k-1}}\big)$$

it follows immediately that $\widehat{\mathbb{P}}_{X_t}(z) = \left(\widehat{\varphi}_{1/n}(z)\right)^n$. where $\widehat{\varphi}_{1/n}$ is the characteristic function of $X_{1/n}$. Let $\mu = \mathbb{P}_{X_1}$, then $\mu = \left(\mathbb{P}_{X_{1/n}}\right)^n$ from which we get $\mathbb{P}_{X_{1/n}} = \mu^{1/n}$. Thus, for any rational number n/m we have $\mathbb{P}_{X_{n/m}} = \mu^{n/m}$.For any irrational number $t > 0$ let r_n any sequence in $\mathbb{Q}$ such that $r_n \to t$. By continuity in probability $X_{r_n} \xrightarrow{\mathbb{P}} X_t$, thus $X_{t_n} \xrightarrow{d} X_t$ and $\mathbb{P}_{X_t} = \mu^t$ follows.

(ii) If μ is infinitely divisible, then μ^t is a distribution with characteristic function $e^{t\log\widehat{\mu}(z)}$. Hence

$$\begin{aligned} \mu^s * \mu^t &= \mu^{s+t} && (17)\\ \mu^0 &= \delta_0 && (18)\\ \mu_t &\to \delta_0 \quad \text{as} \quad t \downarrow 0 && (19) \end{aligned}$$

Now we proceed to construct the corresponding Lévy process in law. Consider $\Omega = \left(\mathbb{R}^d\right)^{[0,\infty)}$ with the Borel product σ–algebra. Consider also the projections $X_t(\omega) = \omega(t)$. For any $n \in \mathbb{Z}_+$ and $0 \le t_0 < t_1 < \ldots < t_n$ define

$$\mu_{t_0\ldots t_n}(B_0 \times \ldots \times B_n) = \int\limits_{(\mathbb{R}^d)^{n+1}} \mathbf{1}_{B_0}(y_0)\mathbf{1}_{B_1}(y_0+y_1)\cdots\mathbf{1}_{B_n}(y_0+\cdots+y_n)\cdot$$
$$\cdot\, \mu^{t_0}(dy_0)\mu^{t_1-t_0}(dy_1)\cdots\mu^{t_n-t_{n-1}}(dy_n)$$

Each $\mu_{t_0\ldots t_n}$ can be extended to $\mathcal{B}\left((\mathbb{R}^d)^{n+1}\right)$ and clearly the family of finite dimensional distributions thus obtained satisfies the Kolmogorov's consistency condition. Therefore, there exists a unique probability measure $\mathbb{P}$ on $\mathcal{F}$ such that

$$\mathbb{P}\left[X_{t_0} \in B_0, \ldots, X_{t_n} \in B_n\right] = \mu_{t_0\ldots t_n}(B_0 \times \ldots \times B_n).$$

In particular, X_t has distribution μ^t. We will show now that $\{X_t : t \ge 0\}$ is indeed a Lévy process in law. If $0 \le t_0 < t_1 < \ldots < t_n$ then for any bounded measurable function $f : \left(\mathbb{R}^d\right)^{n+1} \mapsto \mathbb{R}$

$$\mathbb{E}\left[f(X_{t_0}, \ldots, X_{t_n})\right] = \int_{(\mathbb{R}^d)^{n+1}} f(y_0, y_0+y_1, \ldots, y_0+\cdots+y_n)\cdot$$
$$\cdot\, \mu^{t_0}(dy_0)\mu^{t_1-t_0}(dy_1)\cdots\mu^{t_n-t_{n-1}}(dy_n)$$

In particular, for the function

$$f(x_0, \ldots, x_n) = \exp\left(i\sum_{j=1}^{d}\langle z_j, x_j - x_{j-1}\rangle\right)$$

we get

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{d}\langle z_j, X_j - X_{j-1}\rangle\right)\right] = \prod_{j=1}^{d}\int_{\mathbb{R}^d}\exp\left(i\langle z_j, y_j\rangle\right)\mu^{t_j - t_{j-1}}(dy_j)$$

If follows immediately that $X_j - X_{j-1}$ has law $\mu^{t_j - t_{j-1}}$ and that X_t has independent increments. From (c) we get

$$\mathbb{P}\left[|X_s - X_t| > \varepsilon\right] = \mathbb{P}\left[|X_{|t-s|}| > \varepsilon\right] \to 0 \quad \text{as } s \to t.$$

That is, $\{X_t : t \geq 0\}$ is a Lévy process in law.

(iii) Let $\{X_t\}$ and $\{X'_t\}$ be Lévy processes in law and $X_1 \overset{d}{=} X'_1$. From (i) it follows that $X_{s+t} - X_s \overset{d}{=} X'_{s+t} - X'_s$ for any $s, t \geq 0$. The independent increments property implies that

$$(X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}) \overset{d}{=} (X'_{t_0}, X'_{t_1} - X'_{t_0}, \ldots, X'_{t_n} - X'_{t_{n-1}})$$

for $0 \leq t_0 < \ldots < t_n$. Since $(X_{t_0}, \ldots, X_{t_n})$ is a measurable function of $(X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}})$ then

$$(X_{t_0}, \ldots, X_{t_n}) = (X'_{t_0}, \ldots, X'_{t_n})$$

Lemma 1.3 concludes the proof. □

2.2. **Representation of infinitely divisible distributions.**

Theorem 2.9. *(Lévy–Kintchine)*

(a) *If μ is an infinitely divisible distribution on $\mathbb{R}^d$, then*

$$\widehat{\mu}(z) = \exp\Big[-\tfrac{1}{2}\langle z, Az\rangle + i\langle\gamma, z\rangle + \int_{\mathbb{R}^d}\Big(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle\mathbf{1}_D(x)\Big)\nu(dx)\Big] \tag{20}$$

for $z \in \mathbb{R}^d$, where $D = \{x : |x| \leq 1\}$, A is a symmetric nonnegative–definite $d\times d$ matrix, ν is a measure on $\left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$ satisfying

$$\nu\left(\{0\}\right) = 0 \quad \textit{and} \quad \int_{\mathbb{R}^d}\left(|x|^2 \wedge 1\right)\nu(dx) < \infty, \tag{21}$$

and $\gamma \in \mathbb{R}^d$.

(b) *The representation of $\widehat{\mu}(z)$ in (20) with triplet (A, ν, γ) is unique.*

(c) *Conversely, if A is a symmetric nonnegative–definite $d \times d$ matrix, ν is a measure satisfying (21) and $\gamma \in \mathbb{R}^d$, then there exists an infinitely divisible distribution μ whose characteristic function is given by (20).*

Before we embark ourselves into proving such a result, we introduce first some concepts and ideas that will be at handy later on.
We call (A, ν, γ) as in theorem 2.9, the generating triplet of μ. The A and ν are called respectively, the Gaussian covariance matrix and the Lévy measure of μ. When $A = 0$, μ is said to be purely non Gaussian. When $\nu = 0$, then μ is Gaussian.

The integrand in the right–hand side of (20) is integrable with respect to ν because it is bounded outside any neighborhood of 0, and for fixed z

$$e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbf{1}_D(x) = O(|x|^2) \quad \text{as } |x| \to 0$$

There are many other ways of getting an integrable integrand. Some useful choices are obtained as follows. Let $c : \mathbb{R}^d \mapsto \mathbb{R}$ be a bounded measurable function satisfying

$$\begin{array}{lll} c(x) = 1 + o(|x|) & \text{as} & |x| \to 0 \\ c(x) = O(1/|x|) & \text{as} & |x| \to \infty \end{array} \tag{22}$$

Under these conditions, it is easy to see that for fixed z

$$e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle c(x) = \begin{cases} O(|x|^2) & \text{as } |x| \to 0 \\ O(1) & \text{as } |x| \to \infty \end{cases}$$

On the other hand, equation (20) is now expressed as

$$\begin{aligned} \widehat{\mu}(z) = \exp\Big[& -\tfrac{1}{2}\langle z, Az\rangle + i\langle \gamma_c, z\rangle \\ & + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle c(x)\right) \nu(dx)\Big] \end{aligned} \tag{23}$$

where

$$\gamma_c = \gamma + \int_{\mathbb{R}^d} x \left(c(x) - \mathbf{1}_D(x)\right) \nu(dx) \tag{24}$$

The triplet $(A, \nu, \gamma_c)_c$ is also called generating triplet of ν. Commonly used examples of $c(x)$ are:

(i) $c(x) = \mathbf{1}_{\{|x|\le\varepsilon\}}(x)$ with $\varepsilon > 0$
(ii) $c(x) = \frac{1}{1+|x|^2}$
(iii) $(d = 1)$ $c(x) = \mathbf{1}_{[-1,1]}(x) + \frac{1}{x}\mathbf{1}_{(1,\infty)}(x) - \frac{1}{x}\mathbf{1}_{(-\infty,1)}(x)$
(iv) $(d = 1)$ $c(x) = \frac{\sin x}{x}$

More generally, if $c(x)$ is a measurable function, and if for every z, $e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle c(x)$ is integrable with respect to a given Lévy measure ν, then we obtain (23) with γ_c as in (24). The triplet $(A, \nu, \gamma_c)_c$ is still called generating triplet. If ν satisfies $\int_{|x|\le 1} |x|\, \nu(dx) < \infty$, then letting

$c = 0$ we get the representation

$$\widehat{\mu}(z) = \exp\Big[-\tfrac{1}{2}\langle z, Az\rangle + i\langle \gamma_0, z\rangle + \int_{\mathbb{R}^d} \big(e^{i\langle z,x\rangle} - 1\big)\,\nu(dx)\Big] \tag{25}$$

with $\gamma_0 \in \mathbb{R}^d$. The term γ_0 is called the *drift* of μ. If $\int_{|x|>1}|x|\,\nu(dx) < \infty$, then letting $c = 1$ we have the representation

$$\widehat{\mu}(z) = \exp\Big[-\tfrac{1}{2}\langle z, Az\rangle + i\langle \gamma_1, z\rangle + \int_{\mathbb{R}^d} \big(e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle\big)\,\nu(dx)\Big]. \tag{26}$$

The term γ_1 is called the *center* of μ.

The following analytic proof of theorem 2.9, is taken from [7]. Parts (b) and (c) are proved first by direct methods. Part (a) will follow from a more general result using an integrand as in (23).

Proof. (b) Suppose that $\widehat{\mu}(x)$ is given by (20) with generating triplet (A,ν,γ). The estimate

$$\Big|e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle \mathbf{1}_D(x)\Big| \leq \tfrac{1}{2}|z|^2|x|^2\mathbf{1}_D(x) + 2\mathbf{1}_{D^c}(x)$$

and dominated convergence shows that the expression in the square brackets in (20) is continuous. Therefore $\log\widehat{\mu}(z)$ is well defined. Dominated convergence and the equality

$$\begin{aligned}\log\widehat{\mu}(sz) = &-\tfrac{1}{2}s^2\langle z, Az\rangle + i\langle\gamma, sz\rangle\\ &+ \quad + \int_{\mathbb{R}^d}\big(e^{i\langle sz,x\rangle} - 1 - i\langle sz,x\rangle c(x)\big)\,\nu(dx)\end{aligned}$$

gives

$$\lim_{s\to\infty}\log\widehat{\mu}(sz) = -\tfrac{1}{2}\langle z, Az\rangle. \tag{27}$$

Hence A is uniquely determined by μ. Let $C = [-1,1]^d$ and $\psi(z) = \log\widehat{\mu}(z) + \frac{1}{2}\langle z, Az\rangle$. Equation (27) shows that $\psi(z)$ is completely determined by μ. Clearly

$$\psi(z) - \psi(z+w) = -i\langle\gamma, w\rangle + \int_{\mathbb{R}^d}\Big(e^{i\langle z,x\rangle} - e^{i\langle z+w,x\rangle} + i\langle w,x\rangle\mathbf{1}_D(x)\Big)\,\nu(dx).$$

The estimate

$$\begin{aligned}|e^{i\langle z,x\rangle} - e^{i\langle z+w,x\rangle} + i\langle w,x\rangle| &\leq |1 - e^{i\langle w,x\rangle} + i\langle w,x\rangle| + |\langle w,x\rangle||1 - e^{i\langle z,x\rangle}|\\ &\leq \tfrac{1}{2}|w|^2|x|^2 + |w||z||x|^2\end{aligned}$$

allows us to use Fubini's theorem to get

$$\int_C \psi(z) - \psi(z+w)\,dw = 2^d\int_{\mathbb{R}^d} e^{1\langle z,x\rangle}\left(1 - \prod_{j=1}^{d}\frac{\sin x_j}{x_j}\right)\nu(dx) \tag{28}$$

Let

$$\rho(dx) = \left(1 - \prod_{j=1}^{d} \frac{\sin x_j}{x_j}\right) \nu(dx).$$

Since

$$\prod_{j=1}^{d} \frac{\sin x_j}{x_j} = 1 - \frac{|x|^2}{6} + O(|x|^4) \quad \text{as } |x| \to 0$$

it follows that ρ is a finite measure, and (28) is its Fourier transform. By the Fourier inversion theorem we get that ρ in uniquely determined by ψ, and thus, by μ. That γ is uniquely determined by μ now follows immediately. $\square$

Proof. (c) Given the triplet (A, ν, γ), let $\varphi(z)$ be as in equation (20). For each $n \in \mathbb{N}$, define

$$\varphi_n(z) = \exp\Big[-\tfrac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{|x|>1/n} \Big(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbf{1}_D(x)\Big)\, \nu(dx)\Big]$$

Each measure $\nu_n(\cdot) = \nu\left(\cdot \cap [|x| > 1/n]\right)$ is finite, thus $\varphi_n(z)$ is the characteristic function of the convolution of a Gaussian and a compound Poisson distribution. Since $\varphi_n(z) \to \varphi(z)$ and $\varphi(z)$ is continuous, Lévy continuity and lemma 2.6 imply that $\varphi(z)$ is the characteristic function of some infinitely divisible distribution. $\square$

The proof of (a) will be based on the following result. Let us denote by $f \in C_\sharp$ if $f : \mathbb{R}^d \mapsto \mathbb{R}$ is bounded continuous function vanishing on a neighborhood of 0.

Theorem 2.10. *Let $c : \mathbb{R}^d \mapsto \mathbb{R}$ be a bounded continuous function satisfying (22). Suppose that μ_n, $n \in \mathbb{N}$ are infinitely divisible distributions on $\mathbb{R}^d$ and each $\widehat{\mu}(z)$ has the Lévy–Kintchine representation given by the generating triplet $(A_n, \nu_n, \beta_n)_c$. Let μ be a probability measure on $\mathbb{R}^d$. Then $\mu_n \xrightarrow{d} \mu$ if and only if μ is infinitely divisible and $\widehat{\mu}(z)$ has Lévy–Kintchine representation given by the generating triplet $(A, \nu, \beta)_c$ with A, ν and β satisfying the following conditions*

(i) If $f \in C_\sharp$ then

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x)\nu_n(dx) = \int_{\mathbb{R}^d} f(x)\nu(dx)$$

(ii) Define the symmetric nonnegative matrices $A_{n,\varepsilon}$ by

$$\langle x, A_{n,\varepsilon}\rangle = \langle z, A_n z\rangle + \int_{|x|\le\varepsilon} \langle z, x\rangle^2 \nu_n(dx) \quad n \in \mathbb{N}, \varepsilon > 0.$$

Then

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}|\langle z, A_{n,\varepsilon}z\rangle| = 0$$

(iii) $\beta_n \to \beta$.

The use of continuous $c(x)$ is to compensate for the discontinuity of $\mathbf{1}_D(x)$

Proof. First we proof necessity. If $\mu_n \xrightarrow{d} \mu$ then μ is infinitely divisible, therefore $\widehat{\mu}(z) \neq 0$ for any z and $\widehat{\mu}_n(z) \to \widehat{\mu}(z)$ uniformly in compacta. It follows that $\log\widehat{\mu}_n(z) \to \log\widehat{\mu}(z)$ uniformly in compacta.

Define

$$\rho_n(dx) = \big(|x|^2 \wedge 1\big)\,\nu_n(dx)$$

Claim:

(1) $\sup_n\big\{\rho_n(\mathbb{R}^d)\big\} < \infty$
(2) $\lim\limits_{l\to\infty}\sup_n\int_{|x|>l}\rho_n(dx) = 0$

This means that (ρ_n) is uniformly bounded and uniformly tight. Let us assume for the moment that the claim holds. Then using theorem 1.15 (b) after normalization we can show that there is a subsequence ρ_{n_k} which converges to some finite measure ρ, since we can assume $\inf_n\rho_n(\mathbb{R}^d) > 0$ as the case $\lim_n\rho_n(\mathbb{R}^d) = 0$ is evident. Define ν by setting

(1) $\nu(\{0\}) = 0$
(2) $\nu(dx) = (|x|^2 \wedge 1)^{-1}\,\rho(dx)$ for $|x| > 0$.

In general, ρ might have a point mass at 0, but this is ignored by ν. Let us define the function $g : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{C}$ by

$$g(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle c(x).$$

This function is continuous, and for fixed z it is bounded. Then

$$\begin{aligned}\log\widehat{\mu}_n(z) &= -\tfrac{1}{2}\langle z, A_n z\rangle + i\langle\beta_n, z\rangle + \int_{\mathbb{R}^d}\nu_n(dx)\\ &= -\tfrac{1}{2}\langle z, A_{n,\varepsilon} z\rangle + i\langle\beta_n, z\rangle + I_{n,\varepsilon}(z) + J_{n,\varepsilon}(z) \qquad (29)\end{aligned}$$

where

$$\begin{aligned}I_{n,\varepsilon}(z) &= \int_{|x|\le\varepsilon}\big(g(z,x) + \tfrac{1}{2}\langle\beta_n, z\rangle\big)\big(|x|^2 \wedge 1\big)^{-1}\rho_n(dx)\\ J_{n,\varepsilon}(z) &= \int_{|x|>\varepsilon} g(z,x)\big(|x|^2 \wedge 1\big)^{-1}\rho_n(dx)\end{aligned}$$

Let E be the set of $\varepsilon > 0$ for which $\int_{|x|=\varepsilon} \rho(dx) = 0$. Then by theorem 1.15 (e)

$$\lim_{k\to\infty} J_{n_k,\varepsilon}(z) = \int_{|x|>\varepsilon} g(z,x) \left(|x|^2 \wedge 1\right)^{-1} \rho(dx) \quad \text{for } \varepsilon \in E$$

hence,

$$\lim_{E\ni\varepsilon\downarrow 0} \lim_{k\to\infty} J_{n_k,\varepsilon}(z) = \int_{\mathbb{R}^d} g(z,x)\nu(dx) \tag{30}$$

From lemma 2.3 we get that for fixed z

$$\left(g(z,x) + \tfrac{1}{2}\langle z, \rangle^2\right) \left(|x|^2 \wedge 1\right)^{-1} = o(|x|^2) \quad \text{as } |x| \to 0$$

therefore,

$$\lim_{\varepsilon\downarrow 0} \sup_n |I_{n,\varepsilon}(z)| = 0 \tag{31}$$

Treating real and imaginary part of (29) separately, and combining (30), (31) and the fact the fact that $\log \widehat{\mu}_n(z) \to \log \widehat{\mu}(z)$ uniformly in compacta, we get

$$\lim_{E\ni\varepsilon\downarrow 0} \limsup_{k\to\infty} \langle x, A_{n_k,\varepsilon} z\rangle = \lim_{E\ni\varepsilon\downarrow 0} \liminf_{k\to\infty} \langle x, A_{n_k,\varepsilon} z\rangle \tag{32}$$

$$\limsup_{k\to\infty} \langle \beta_{n_k}, z\rangle = \liminf_{k\to\infty} \langle \beta_{n_k}, z\rangle \tag{33}$$

both sides of both (32) and (33) being finite. It follows that there exist β such that $\beta_{n_k} \to \beta$. Since each side of (32) is a nonnegative quadratic form on z, it is equal to $\langle z, Az\rangle$ form some symmetric, nonnegative–definite matrix A. We can drop the restriction $\varepsilon \in E$ in (32) since $\langle z, A_{n,\varepsilon} z\rangle$ is monotone increasing in ε. We have shown that

$$\widehat{\mu}(z) = \exp\Big[-\tfrac{1}{2}\langle z, Az\rangle + i\langle \beta, z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle c(x)\right) \nu(dx)\Big]$$

and that (i), (ii) and (iii) hold along a subsequence (μ_{n_k}). By part (c) of theorem 2.9, the triplet $(A, \nu, \beta)_c$ is unique. Since we could have started the argument with any subsequence of (μ_n), this uniqueness ensures that (i), (iii) and

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \langle x, A_{n,\varepsilon} z\rangle = \lim_{\varepsilon\downarrow 0} \liminf_{n\to\infty} \langle x, A_{n,\varepsilon} z\rangle \tag{34}$$

It is no hard to check that (34) is equivalent to (ii).
Now we turn to the claim. Let $C(h)=[-h,h]^d$. By Fubini's theorem

$$\begin{aligned} -\int_{C(h)} &\log\widehat{\mu}_n(z)\,dz \\ &= \tfrac{1}{2}\int_{C(h)}\langle z, A_n\rangle\,dz - \int_{\mathbb{R}^d}\int_{C(h)} g(z,x)\,dz\,\nu_n(dx) \\ &\geq (2h)^d\int_{\mathbb{R}^d}\left(1-\prod_{j=1}^d \tfrac{\sin hx_j}{hx_j}\right)\nu_n(dx). \end{aligned} \tag{35}$$

Since

$$\Lambda = \inf_{x\in\mathbb{R}^d}\left(1-\prod_{j=1}^d \tfrac{\sin hx_j}{hx_j}\right)\left(|x|^2\wedge 1\right)^{-1} > 0$$

then

$$-\int_{C(1)}\log\widehat{\mu}_n(z)\,dz \geq 2^d\Lambda\int_{\mathbb{R}^d}\rho_n(dx).$$

In particular, since $\log\widehat{\mu}_n(z)\to\log\widehat{\mu}(z)$ uniformly in $C(1)$, it follows that $\sup_n\{\rho_n(\mathbb{R}^d)\}<\infty$. On the other hand, continuity of $\log\widehat{\mu}(z)$ implies that

$$\lim_{h\to 0} -\tfrac{1}{(2h)^d}\int_{C(h)}\log\widehat{\mu}(z)\,dz = -\log\widehat{\mu}(0) = 0$$

Hence, for every $\varepsilon>0$ there are n_0 and $h_0>0$ such that

$$\int_{\mathbb{R}^d}\left(1-\prod_{j=1}^d \tfrac{\sin h_0x_j}{h_0x_j}\right)\nu_n(dx) < \tfrac{\varepsilon}{2} \quad \text{for } n\geq n_0$$

If $|x|>2\sqrt{d}/h_0$, then $|x_{j_0}|>2/h_0$ for some j_0 and

$$1-\prod_{j=1}^d \tfrac{\sin h_0x_j}{h_0x_j} \geq 1-\left|\tfrac{\sin h_0x_{j_0}}{h_0x_{j_0}}\right| \geq 1-\tfrac{1}{h_0|x_{j_0}|} > \tfrac{1}{2}$$

Hence, for every $n\geq n_0$

$$\int\limits_{|x|>2\sqrt{d}/h_0}\rho_n(dx) = \int\limits_{|x|>2\sqrt{d}/h_0}(|x|^2\wedge 1)\nu_n(dx) < \varepsilon$$

Now we prove the sufficiency. Define $\rho_n(dx)=(|x|^2\wedge 1)\nu_n(dx)$ and $\rho(dx)=(|x|^2\wedge 1)\nu(dx)$ as above. Let E be as before. From (i) we get (30). Since

$$\int_{|x|\leq\varepsilon}\langle z,x\rangle^2\nu_n(dx) \leq \langle z, A_{n,\varepsilon}z\rangle$$

conditions (i) and (ii) imply that $\big(\rho_n(\mathbb{R}^d)\big)$ is uniformly bounded. Therefore (31) holds as well. Using equation (29) and conditions (ii) and (iii) we obtain

$$\lim_{n\to\infty} \log \widehat{\mu}_n(z) = \tfrac{1}{2}\langle z, Az\rangle + i\langle \beta, z\rangle + \int_{\mathbb{R}^d} g(z,x)\nu(dx)$$

The right–hand side equals to $\log \widehat{\mu}(x)$. Therefore $\mu_n \xrightarrow{d} \mu$. □

Proof of theorem 2.9(a) Given the infinitely divisible distribution μ, choose any sequence $t_n \downarrow 0$. Define μ_n by

$$\widehat{\mu}_n(z) = \exp\left[t_n^{-1}(\widehat{\mu}^{t_n}(z) - 1)\right] = \exp\left[t_n^{-1}\int_{\mathbb{R}^d\setminus\{0\}} (e^{1\langle z,x\rangle} - 1)\mu^{t_n}(dx)\right]$$

The distribution of μ_n is compound Poisson. Note that

$$\widehat{\mu}_n(z) = \exp\left[t_n^{-1}(e^{t_n \log \widehat{\mu}(z)} - 1)\right] = \exp\left[t_n^{-1}(t_n \log \widehat{\mu}(z) + O(t_n^2))\right]$$

for each z as $n \to \infty$. Hence $\widehat{\mu}_n(z) \to e^{\log \widehat{\mu}(z)} = \widehat{\mu}(z)$. Since μ_n has the representation (23), theorem 2.10 can be applied to conclude that $\widehat{\mu}(z)$ has the Lévy–Kintchine representation with triplet $(A, \nu, \beta)_c$. This representation can be written in unique way in form (20). □

The following results are immediate consequences of the proof of part (a)

Corollary 2.11. *Every infinitely divisible distribution is the limit in law of a sequence of compound Poisson distributions.*

Corollary 2.12. *Let $t_n \downarrow 0$. If ν is the Lévy measure of an infinitely divisible distribution μ, then for any $f \in C_\sharp$*

$$t_n^{-1}\int_{\mathbb{R}^d\setminus\{0\}} f(x)\mu^{t_n}(dx) \to \int_{\mathbb{R}^d} f(x)\nu(dx)$$

Proof. Take $\nu_n(\cdot) = t_n^{-1}\mu^{t_n}(\cdot \cap \mathbb{R}^d \setminus \{0\})$, and apply theorem 2.10(i). □

Corollary 2.13. *If μ has generating triplet (A, ν, γ), then μ^t has generating triplet $(tA, t\nu, t\gamma)$*

2.3. **Additive Processes.** Indefinitely divisible distributions have close connection not only with Lévy processes but also with additive processes.

Theorem 2.14. *If $\{X_t : t \geq 0\}$ is an additive process in law on $\mathbb{R}^d$, then for every t, the law of X_t is infinitely divisible.*

The proof of the previous theorem is rather easy once we have at our disposal the following result that is related to null arrays. This is one of the fundamental theorems on sums of independent variables.

Theorem 2.15. *(Kintchine–Kolmogorov) Let S_{nk} null array on $\mathbb{R}^d$ with row sums S_n, If for some $b_n \in \mathbb{R}^d$, $n = 1, \ldots$, the distribution of $S_n - b_n$ converges to a distribution μ, then μ is infinitely divisible.*

The following lemmas will be useful for the proof of theorem 2.15. The first one introduces some some kind of centering for random variables. The second gives some estimate for the characteristic function of a random variable that is in some sense centered.

Define the map $\tau_j : \mathbb{R}^d \mapsto \mathbb{R}^d$ whose components are given by

$$\tau_j(x) = \tau(x_1, \ldots, x_d) = x_j \mathbf{1}_{[-1,1]}(x_j) + \mathbf{1}_{(1,\infty)}(x_j) - \mathbf{1}_{(\infty,-1)}(x_j)$$

Lemma 2.16. *For any random variable X on $\mathbb{R}^d$ with law μ there exists $a \in \mathbb{R}^d$ such that*

$$\mathbb{E}\left[\tau(X - a)\right] = 0 \tag{36}$$

Proof. Note that the function $f(a) = \mathbb{E}[\tau_j(X - a)]$ depends only on a_j and as such, it is continuous and decreasing. It tends to 1 and -1 as a_j tends to $-\infty$ and ∞ respectively. Therefore it vanishes at some point a_j. Repeating the argument component by component we obtain the point a. □

Lemma 2.17. *Let X be a random variable on $\mathbb{R}^d$ with law μ and such that $\mathbb{E}[\tau(X)] = 0$. Then*

$$|\widehat{\mu}(z) - 1| \leq \left(2 + \sqrt{d}\,|z| + \tfrac{1}{2}|z|^2\right) \mathbb{E}\left[|\tau(X)|^2\right] \quad \text{for } z \in \mathbb{R}^d \tag{37}$$

Proof. Notice that $\widehat{\mu}(z) - 1 = \mathbb{E}\left[e^{i\langle z, X\rangle} - 1 - i\langle z, \tau(X)\rangle\right]$. For x in the box $C(1)$ we get

$$|e^{i\langle z,x\rangle} - 1 - i\langle z, \tau(x)\rangle| \leq \tfrac{1}{2}|z|^2|\tau(x)|^2 \tag{38}$$

For x outside the unit box we have that $|x_k| > 1$ for some k. thus

$$|e^{i\langle z,x\rangle} - 1 - i\langle z, \tau(x)\rangle| \leq 2 + \sqrt{d}\,|z| \leq (2 + \sqrt{d}\,|z|)|\tau(x)|^2$$

Pulling this two cases together gives (37). □

Proof of theorem 2.15. For each $n \in \mathbb{N}$ and $k = 1, \ldots, r_n$ choose $a_{nk} \in \mathbb{R}^d$ such that $\mathbb{E}\left[\tau(S_{nk} - a_{nk})\right] = 0$. Let us denote by μ_{nk} their respective distributions. Also, let

$$\gamma_n = \sum_{k=1}^{r_n} a_{nk} - b_n.$$

We will show that

$$\exp\left[i\langle\gamma_n, z\rangle + \sum_{k=1}^{r_n}\left(\widehat{\mu}_{nk}(z) - 1\right)\right] \to \widehat{\mu}(z), \quad n \to \infty \tag{39}$$

This will prove the theorem since the expression on the right–hand side of the limit above is the characteristic function of a delta and a compound Poisson distribution.
The estimate

$$\left|\widehat{\mathbb{P}}_{S_{nk}}(z) - 1\right| = \left|\int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1\right)\mathbb{P}_{S_{nk}}(dx)\right| \le \varepsilon|z| + 2\mathbb{P}_{S_{nk}}\left[|S_{nk}| > \varepsilon\right]$$

together with the fact that S_{nk} is a null array, imply that for any compact set $K \subset \mathbb{R}^d$

$$\sup_{z\in K}\max_{1\le k\le n}\left|\widehat{\mathbb{P}}_{nk}(z) - 1\right| \to 0, \quad n \to \infty \tag{40}$$

Our goal now is to show that the $S_{nk} - a_{nk}$ form a null array as well. To this end, we first show that

$$\max_{1\le k\le n}|a_{nk}| \to 0, \quad n \to \infty \tag{41}$$

Fix any $0 < \varepsilon < 1$ and let $\bar{\varepsilon}$ be the vector whose components are all ε. Let $\bar{a} \in \mathbb{R}^d$ is such that $a_j \le -\varepsilon$, and $\bar{b} \in \mathbb{R}^d$ such that $b_j \ge \varepsilon$. Splitting the space Ω as

$$\left[S_{nk}^{(j)} \le -\tfrac{\varepsilon}{2}\right]\bigcup\left[\left|S_{nk}^{(j)}\right| < \tfrac{\varepsilon}{2}\right]\bigcup\left[S_{nk}^{(j)} > \tfrac{\varepsilon}{2}\right]$$

we obtain for all n large enough

$$\begin{aligned}
\mathbb{E}\left[\tau_j(S_{nk} - \bar{a})\right] &\ge \mathbb{E}\left[\tau_j(S_{nk} + \bar{\varepsilon})\right] \\
&\ge \mathbb{E}\left[\tau_j(S_{nk} + \bar{\varepsilon}); \left|S_{nk}^{(j)}\right| < \tfrac{\varepsilon}{2}\right] + \mathbb{E}\left[\tau_j(S_{nk} + \bar{\varepsilon}); S_{nk}^{(j)} < -\tfrac{\varepsilon}{2}\right] \\
&\ge \tfrac{\varepsilon}{2}\mathbb{P}\left[\left|S_{nk}^{(j)}\right| < \tfrac{\varepsilon}{2}\right] - \mathbb{P}\left[S_{nk}^{(j)} < -\tfrac{\varepsilon}{2}\right] \\
&\ge \tfrac{\varepsilon}{2}\mathbb{P}\left[\left|S_{nk}^{(j)}\right| < \tfrac{\varepsilon}{2}\right] - \mathbb{P}\left[\left|S_{nk}^{(j)}\right| > \tfrac{\varepsilon}{2}\right] \\
&\ge \tfrac{\varepsilon}{2}(1 - \tfrac{\varepsilon}{8}) - \tfrac{\varepsilon}{8} > \tfrac{\varepsilon}{4} - \tfrac{\varepsilon}{8} > 0
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathbb{E}\left[\tau_j(S_{nk}-\bar{b})\right] &\leq \mathbb{E}\left[\tau_j(S_{nk}-\bar{\varepsilon})\right]\\
&\leq \mathbb{E}\left[\tau_j(S_{nk}-\bar{\varepsilon}); \left|S^{(j)}_{nk}\right|<\tfrac{\varepsilon}{2}\right]+\mathbb{E}\left[\tau_j(S_{nk}-\bar{\varepsilon}); S^{(j)}_{nk}>\tfrac{\varepsilon}{2}\right]\\
&\leq -\tfrac{\varepsilon}{2}\mathbb{P}\left[\left|S^{(j)}_{nk}\right|<\tfrac{\varepsilon}{2}\right]+\mathbb{P}\left[S^{(j)}_{nk}>\tfrac{\varepsilon}{2}\right]\\
&\leq -\tfrac{\varepsilon}{2}\mathbb{P}\left[\left|S^{(j)}_{nk}\right|<\tfrac{\varepsilon}{2}\right]+\mathbb{P}\left[\left|S^{(j)}_{nk}\right|>\tfrac{\varepsilon}{2}\right]\\
&\leq -\tfrac{\varepsilon}{2}(1-\tfrac{\varepsilon}{8})+\tfrac{\varepsilon}{8}<-\tfrac{\varepsilon}{4}+\tfrac{\varepsilon}{8}<0
\end{aligned}$$

This shows (41), from which it follows

$$\max_{1\leq k\leq r_n}\mathbb{P}\left[|S_{nk}-a_{nk}|>\varepsilon\right]\to 0,\quad n\to\infty.$$

Thus, for any compact $K\subset\mathbb{R}^d$

$$\sup_{z\in K}\max_{1\leq k\leq r_n}|\widehat{\mu}_{nk}(z)-1|\to 0,\quad n\to\infty \tag{42}$$

Hence for n sufficiently large the right–hand side of (42) is less than $1/2$. Thus for a fixed compact set K, the continuous logarithm of $\widehat{\mu}_{nk}(z)$, $\log\widehat{\mu}_{nk}(z)$, is defined as the principal branch of the analytic logarithm, ln, of $\widehat{\mu}_{nk}(z)$. Then for $z\in K$

$$\log\widehat{\mu}_{nk}(z)=\ln(\widehat{\mu}_{nk}(z))=\ln(1+\theta_{nk}(z))=\theta_{nk}(z)(1+\rho_{nk}(z)) \tag{43}$$

where

$$\theta_{nk}(z)=\widehat{\mu}_{nk}(z)-1,\quad \rho_{nk}(z)=\sum_{m=1}^{\infty}\tfrac{(-1)^m}{m+1}\theta^m_{nk}(z)$$

It follows from (42) and the estimate

$$|\rho_{nk}(z)|\leq|\theta_{nk}(z)|\,/\,(1-|\theta_{nk}(z)|)$$

that

$$\sup_{z\in K}\max_{1\leq k\leq r_n}|\rho_{nk}(z)|\to 0,\quad n\to\infty \tag{44}$$

Let

$$v_n=\sum_{k=1}^{r_n}\mathbb{E}\left[|\tau(S_{nk}-a_{nk})|^2\right].$$

By lemma 2.17 we get

$$\sum_{k=1}^{r_n}|\theta_{nk}(z)|\leq\left(2+\sqrt{d}\,z+\tfrac{1}{2}|z|^2\right)v_n\qquad\text{for}\quad z\in K \tag{45}$$

Combining (43), (44) and (45) we obtain

$$\sum_{k=1}^{r_n} \log \widehat{\mu}_{nk}(z) = \sum_{k=1}^{r_n} (\widehat{\mu}_{nk}(z) - 1) + o(v_n) \tag{46}$$

uniformly on $z \in K$ as $n \to 0$. Next we show that

$$v_n = O(1), \quad \text{as} \quad n \to \infty \tag{47}$$

which would imply that $o(v_n) = o(1)$. Set $K = \{z : |z| \leq 1\}$ and consider the cube $C(h)$ contained in K with h small enough so that $|\hat{(\mu)}(z) - 1| < 1/2$. By hypothesis and theorem 1.16(vi) we get

$$e^{i\langle \gamma_n, z\rangle} \prod_{k=1}^{r_n} \widehat{\mu}_{nk}(z) \to \widehat{\mu}(z) \tag{48}$$

uniformly on compact sets. Thus, uniformly on $C(h)$ we have

$$i\langle \gamma_n, z\rangle + \sum_{k=1}^{r_n} \log \widehat{\mu}_{nk}(z) \to \log \widehat{\mu}(z), \quad n \to \infty$$

Using, (46), we integrate the expression above over $C(h)$. Because of (38) we can apply Fubini's theorem, which leads to

$$\sum_{k=1}^{r_n} \int_{\mathbb{R}^d} \left(1 - \prod_{j=1}^{d} \frac{\sin hx_j}{hx_j}\right) \mu_{nk}(dx) + o(v_n) + o(1) = c_1$$

as $n \to \infty$, where

$$c_1 = -\tfrac{1}{(2d)^d} \int_{C(h)} \log \widehat{\mu}(z)\, dz$$

It follows immediately that

$$\begin{aligned} |c_1| &\geq \Lambda \sum_{k=1}^{r_n} \int_{\mathbb{R}^d} |\tau(x)|^2 \mu_{nk}(dx) + o(v_n) + o(1) \\ &= \Lambda v_n + o(v_n) + o(1) \end{aligned}$$

where

$$0 < \Lambda = \inf_{x \in \mathbb{R}^d} \left(1 - \prod_{j=1}^{d} \frac{\sin hx_j}{hx_j}\right) \tfrac{1}{|x|^2 \wedge 1} < \infty.$$

This proves (47). Pulling (46), (47) and (48) together we obtained that

$$\mu(z) = \exp\left[i\langle \gamma_n, z\rangle + \sum_{k=1}^{r_n} (\widehat{\mu}_{nk}(z) - 1) + o(1)\right] + o(1)$$

as $n \to \infty$ on the set K. This is precisely equation (39) and the proof is complete . □

Proof of theorem 2.14. Fix $t > 0$ and introduce the points $t_{n,k} = kt/n$ for each $n \in \mathbb{N}$ and $k = 1, \dots, n$. Define $S_{nk} = X_{t_{n,k}} - X_{t_{n,k-1}}$. Lemma 1.9 implies that the S_{nk} form a null array. The row sum S_n equals X_t for each n. Applying theorem 2.15 with $\mu = \mathbb{P}_{X_t}$ and $b_n = 0$ concludes the proof. □

The following result gives the existence of additive processes in law.

Theorem 2.18. *(Existence of additive processes)*

(i) Let $\{X_t : t \geq 0\}$ and additive process in law on $\mathbb{R}^d$. For $0 \leq s \leq t < \infty$ define the system of distributions $\mu_{s,t}$ as the law of $X_t - X_s$. Then each $\mu_{s,t}$ is infinitely divisible and

$$\mu_{s,t} * \mu_{t,u} = \mu_{s,u} \quad \textit{for} \quad 0 \leq s \leq t < \infty \tag{49}$$

$$\mu_{s,s} = \delta_0 \quad \textit{for} \quad 0 \leq s \tag{50}$$

$$\mu_{s,t} \to \delta_0 \quad \textit{as} \quad s \to t \tag{51}$$

(ii) Conversely, if $\{\mu_{s,t} : 0 \leq s \leq t < \infty\}$ is a system of probability measures on $\mathbb{R}^d$ satisfying (49)–(51) then, there exists an additive process in law $\{X_t : t \geq 0\}$ such that for $0 \leq s \leq t < \infty$, $X_t - X_s$ has law $\mu_{s,t}$.

(iii) If $\{X_t\}$ and $\{X'_t\}$ are additive processes in law on $\mathbb{R}^d$ such that $X_t \stackrel{d}{=} X'_t$ for each $t \geq 0$, then $X \stackrel{d}{=} X'$.

Proof. (i) For $s \geq 0$ fixed, $\{X_{t+s} - X_s : t \geq 0\}$ is an additive process in law. Hence, from theorem 2.14, $\mu_{s,t}$ is infinitely divisible. Property (49) comes from independence of increments and $X_u - X_s = (X_u - X_t) + (X_t - X_s)$. Properties (50) and (51)) follow from the stochastic continuity of X.

(ii) The same proof as for theorem 2.8 works if we replace conditions (17)–(19) by conditions (49)–(51), and μ^{t-s} by $\mu_{s,t}$.

(iii) Let $\{X_t\}$ and $\{X'_t\}$ be additive process in law such that $X_t \stackrel{d}{=} X'_t$ for all $t \geq 0$. Let $\mu_{0,t}$ and $\mu'_{0,t}$ be the laws of X_t and X'_t respectively. Then $\mu_{0,t} = \mu'_{0,t}$. Since $\mu_{0,t}$ is infinitely divisible we have that $\widehat{\mu}_{s,t}(z) \neq 0$ for any z. Therefore, from $\mu_{0,t} = \mu_{0,s} * \mu_{s,t}$ and $\mu'_{0,t} = \mu'_{0,s} * \mu'_{s,t}$ it follows that $\mu_{s,t} = \mu'_{s,t}$. The rest of the proof is just as in theorem 2.8(iii). □

The following result is in the same spirit as theorem 2.18 but is given in terms of the Lévy–Kintchine characterization of infinitely divisible distributions.

Theorem 2.19. *(i) Suppose $\{X_t : t \geq 0\}$ is an additive process in law on $\mathbb{R}^d$. For each $t \geq 0$, let (A_t, ν_t, γ_t) be the generating triplet of the infinitely divisible distribution $\mu_t = \mathbb{P}_{X_t}$. Then the following conditions are satisfied.*

(a) $A_0 = 0, \nu_0 = 0, \gamma_0 = 0$
(b) If $0 \le s \le t \le \infty$, *then* $\langle z, A_s z\rangle \le \langle z, A_t z\rangle$, *and* $\nu_s(B) \le \nu_t(B)$ *for each* $B \in \mathcal{B}(\mathbb{R}^d)$.
(c) As $s \to t$ *in* $[0,\infty)$ *we have* $\langle z, A_s z\rangle \to \langle z, A_t z\rangle$, $\nu_s(B) \to \nu_t(B)$ *for any* $B \in \mathcal{B}(\mathbb{R}^d)$ *such that* $dist(0, B) > 0$, *and* $\gamma_s \to \gamma_t$.

(ii) Let $\{\mu_t, t \ge 0\}$ *be a system of infinitely divisible probability measures on* $\mathbb{R}^d$ *with generating triplets* (A_t, ν_t, γ_t) *satisfying conditions (a)–(c). Then there exists, uniquely up to identity in law, an additive process in law* $\{X_t : t \ge 0\}$ *on* $\mathbb{R}^d$ *such that* $\mathbb{P}_{X_t} = \mu_t$.

Proof. (i) The process $\{X_t\}$ determines a system of measures $\{\mu_{s,t} : 0 \le s \le t \le \infty\}$ as in theorem 2.18. Property (a) is obvious and (b) follows from the fact that $\mu_t = \mu_{0,t}$ and $\mu_{0,t} = \mu_{0,s} * \mu_{s,t}$. As for property (c), take a sequence s_n that converges to t, and let $t_0 = t \vee \sup_n s_n$. The stochastic continuity of X_t implies that $\mu_{s_n} \xrightarrow{d} \mu_t$ and by theorem 2.10 (ii) and (iii) we obtain that

$$\gamma_{s_n} \to \gamma_t \quad \text{as} \quad n \to 0$$

and

$$\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} \left| \langle z, A_{s_n} z\rangle + \int\limits_{|x|<\varepsilon} \langle z, x\rangle^2 \nu_{s_n}(dx) - \langle z, A_t z\rangle \right| = 0$$

Property (b) and dominated convergence gives

$$0 \le \int\limits_{|x|<\varepsilon} \langle z, x\rangle^2 \nu_{s_n}(dx) \le \int\limits_{|x|<\varepsilon} \langle z, x\rangle^2 \nu_{t_0}(dx) \to 0 \quad \text{as} \quad \varepsilon \downarrow 0$$

from where it follows that

$$\lim_{n\to\infty} \langle z, A_{s_n} z\rangle = \langle z, A_t z\rangle$$

For any $s \le t_0$, $\nu_{s_n} \le \nu_{t_0}$, hence by the Radon–Nikodym theorem we have that $\nu_s(dx) = g_s(x)\nu_{t_0}(dx)$ with $0 \le g_s(x) \le 1$. Moreover, if $s < s' \le t_0$, then $g_s(x) \le g_{s'}(x)$ ν_{t_0}–a.e. If $s_n \uparrow t$ then by monotone convergence and theorem 2.10 (i) we have that $g_{s_n}(x) \uparrow g_t(x)$ as $n \to \infty$ ν_{t_0}–a.s. Hence, for any $\varepsilon > 0$, $\nu_{s_n}(B) \to \nu_t(B)$ for any Borel set $B \subset \{x : |x| > \varepsilon\}$.

(ii) We define, for each $0 \le s \le t < \infty$, the infinitely divisible distribution $\mu_{s,t}$ as the one with generating triplet $(A_t - A_s, \nu_t - \nu_t, \gamma_t - \gamma_s)$. Thus, $\mu_t = \mu_{0,t}$. We need to show that these system of distributions satisfy conditions (49)–(51) of theorem 2.15. Among them, (49) and (50) are clear. As for (51), it follows from (b) and (c) using similar arguments as in the proof of the first part of the present theorem. □

The following remarks will be useful for the Lévy–Itô decomposition theorem.

Remark 2.1. Given an additive process in law, $\{X_t : t \geq 0\}$, with generating triplet (A_t, ν_t, γ_t) we define $\tilde{\nu}([0,t] \times B) = \nu_t(B)$ for $t \geq 0$ and B a box in $\mathbb{R}^d$. A usual extension argument in measure theory warranties the existence of a unique measure $\tilde{\nu}$ on $\mathcal{B}\left([0,\infty) \times \mathbb{R}^d\right)$ such that

$$\tilde{\nu}([0,t] \times B) = \nu_t(B) \tag{52}$$

for $t \geq 0$ and $B \in \mathcal{B}(\mathbb{R}^d)$. This measure $\tilde{\nu}$ has the following properties

$$\tilde{\nu}([0,\infty) \times \{0\}) = 0 \tag{53}$$

$$\tilde{\nu}(\{t\} \times \mathbb{R}^d) = 0 \tag{54}$$

$$\int\limits_{[0,t]\times\mathbb{R}^d} (1 \wedge |x|^2)\tilde{\nu}(d(t,x)) = \int\limits_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu_t(dx) < \infty \tag{55}$$

for $t \geq 0$. Conversely, if a measure $\tilde{\nu}$ on $[0,\infty) \times \mathbb{R}^d$ satisfies (53)–(55), then the measures defined by (52) satisfy conditions (a)–(c) of theorem 2.19(i).

In the Lévy–Itô characterization theorem we will consider measures $\tilde{\nu}$ on $(0,\infty) \times (\mathbb{R}^d \setminus \{0\})$. If the corresponding process $\{X_t : t \geq 0\}$ is a Lévy process in law, then $\tilde{\nu}$ is the product of the Lebesgue measure on $[0,\infty)$ and the Lévy measure of X_1.

To conclude this section, we prove that an additive process in law $\{X_t : t \geq 0\}$ has a cadlag version, i.e., there is a right–continuous with left–limits additive process $\{Y_t : t \geq 0\}$ such that

$$\mathbb{P}\left[X_t \neq Y_t\right] = 0 \quad \text{for each}\, t \geq 0.$$

To this end, we will diverge from the approach followed in [7] and give instead a probabilistic proof of this fact, much more in the spirit of [6]. Theorem 1.10 and the following lemma will be needed for our purpose.

Lemma 2.20. *(Convergence in $\mathbb{R}^d$) Let (a_n) be a sequence in $\mathbb{R}^d$. Then a_n converges if and only if $e^{i\langle u, a_n\rangle}$ converges for almost every $u \in \mathbb{R}^d$.*

Proof. Only sufficiency needs to be proved. Consider a standard normal vector η in $\mathbb{R}^d$. If we assume $e^{i\langle u, a_n\rangle}$ converges almost everywhere, then by dominated convergence we get that $\mathbb{E}\left[\exp\left(i\langle t, \eta(a_n - a_m)\rangle\right)\right] \to 1$ as $m, n \to \infty$. Theorems 1.15(d) and 1.16(vii) show that $\eta(a_n - a_m) \xrightarrow{\mathbb{P}} 0$

as $n, m \to \infty$, where $\mathbb{P}$ is the standard Gaussian measure in $\mathbb{R}^d$. Thus a_n is a Cauchy sequence and therefore convergent. $\square$

Theorem 2.21. *Let the process X in $\mathbb{R}^d$ be an additive process in law. Then X has a rcll modification with out fixed jumps.*

Proof. Theorem 2.10 says that $\varphi_t(z) = \widehat{\mathbb{P}}_{X_t}(z)$ is infinitely divisible and thus it never vanishes. We may define for each $u \in \mathbb{R}^d$ and $t \geq 0$

$$M_t^u = \frac{e^{i\langle u, X_t\rangle}}{\varphi_t(u)}$$

Clearly this is a continuous in probability martingale in t for each u. Theorem 1.10 implies that there is $\Omega_u \subset \Omega$ with $\mathbb{P}[\Omega_u] = 1$ where $e^{i\langle u, X_t\rangle}$ is rcll. Restating the definition of Ω_u in terms of up-crossings, we note that the set $A = \{(u, \omega) : u \in \mathbb{R}^d, \omega \in \Omega_u\} = \{(u, \omega) : u \in \mathbb{Q}^d, \omega \in \Omega_u\}$ and hence, it is product measurable in $\big(\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}\big)$. Writing $A_\omega = \{u : (u, \omega) \in A\}$, we have by Fubini's lemma that

$$(\lambda^d \otimes \mathbb{P})[A^c] = \mathbb{E}\left[\lambda^d(A^c_\omega)\right] = 0.$$

If follows that $\Omega' = \{\omega : \lambda(A^c_\omega) = 0\}$ has probability 1 and if $\omega \in \Omega'$, then $u \in A_\omega$ almost everywhere. Lemma 2.20 shows that X_t is rcll and corollary to theorem 1.10 implies that there are no fixed jumps. $\square$

3. The Lévy–Itô decomposition of sample paths of additive processes

The Lévi–Itô decomposition expresses paths of additive processes as a sum of two independent parts: a continuous part and a part given by a compensated sum of jumps. The two proofs will be presented. The first one follows [7] and is an analytic proof based on the Lévy–Kintchine representation formula. The second one follows [6] and is a probabilistic proof, the Lévy–Kintchine formula results as a consequence.

3.1. **Poisson point processes.**

Now we introduce the concept of *Poisson point process* which will provide the necessary tools to formulate and prove the Lévy–Itô decomposition.

Let $(\Psi, \mathcal{B})$ be a measurable space and consider the set $\mathcal{M}$ of all measures defined on that space. We can equip $\mathcal{M}$ with a σ–algebra, $\Sigma_{\mathcal{M}}$, generated by the functions $X_A : \mathcal{M} \mapsto [0, \infty]$ defined for each $A \in \mathcal{B}$ by $\mu \mapsto \mu(A)$. Very often it is more convenient to consider a smaller family of measures on $(\Psi, \mathcal{B})$. An instance of this situation might occur when there is a collection $\mathcal{C}$ closed under finite intersections with $\mathcal{B} = \sigma(\mathcal{C})$ and such that Ψ can be expressed as a countable union of sets in $\mathcal{C}$. Then it is of interest to consider the family $\mathcal{M}$ of all measures which are finite on each $A \in \mathcal{C}$. To obtain a useful measurable space, then we define $\Sigma_{\mathcal{M}}$ to be the σ–algebra in $\mathcal{M}$ generated by the maps X_A with $A \in \mathcal{C}$. The best example of this situation arises when we deal with locally compact Polish space Ψ together with its the Borel σ–algebra. Then we restrict our attention to the family $\mathcal{M}$ of *Radon measures*, i.e, measures which are finite on any compact set. Then we set $\mathcal{C}$ to be the collection of all measurable relatively compact subsets of Ψ. It is easy to show that in this case, the subset $\mathcal{Z}(\mathcal{M})$ of all $\overline{Z}_+$–valued measures in $\mathcal{M}$ is $\Sigma_{\mathcal{M}}$–measurable.

Definition 3.1. Let $(\Psi, \mathcal{B})$ be a σ–space, and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. A *random measure* X is a measurable function $X : (\Omega, \mathcal{F}) \mapsto (\mathcal{M}, \Sigma_{\mathcal{M}})$. A random measure X is a *point process* if X takes values on $\mathcal{Z}(\mathcal{M})$.
If X is a point process on $(\Psi, \mathcal{B})$, then the map $B \mapsto \mathbb{E}[X(B)]$ is the *intensity measure* of X

Lemma 3.1. *(Uniqueness of random measures) Let X and Y be random measures on some locally compact Polish space Φ defined in the probability spaces $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ respectively. Then the following are equivalent:*

(a) $X \overset{d}{=} Y$

(b) For any $f \in \mathcal{C}^+_{00}(\Phi)$,

$$\int_{\Omega_1} \left[e^{-\int_\Phi f(\phi,\omega_1) X(d\phi,\omega_1)} \right] \mathbb{P}_1(d\omega_1) = \int_{\Omega_2} \left[e^{-\int_\Phi f(\phi,\omega_2) X(d\phi,\omega_2)} \right] \mathbb{P}_2(d\omega_2)$$

(c) For any $n \in \mathbb{N}$ and measurable relatively compact sets $B_1, \ldots, B_n$,

$$(X(B_1), \ldots, X(B_n)) \overset{d}{=} (Y(B_1), \ldots, Y(B_n))$$

Proof. The equivalence of *(a)* and *(c)* follows from Lemma 1.3 noting that a random measure can be thought as a stochastic process with index $\mathcal{B}(\Phi)$.
Clearly (c) implies (b).
Assuming *(b)*, then since $\mathcal{C}^+_{00}(\Phi)$ is closed under positive linear combinations,

$$\left(\int_\Phi f_1(\phi) X(dx,), \ldots, \int_\Phi f_n(\phi) X(d\phi,) \right) \\ \overset{d}{=} \left(\int_\Phi f_1(\phi) Y(dx,), \ldots, \int_\Phi f_n(\phi) Y(d\phi,) \right)$$

Then we get $\mathbb{P}_1 \circ X^{-1} = \mathbb{P}_2 \circ Y^{-1}$ on $\mathcal{F} = \sigma\left(p_f : f \in \mathcal{C}_{00}^+\right)$ where $p_f : \mu \mapsto \int f(x)\,\mu(dx)$, and it remains to show that $\mathcal{F}$ contains

$$\mathcal{G} \equiv \sigma\left(p_B : B \text{ measurable relatively compact}\right)$$

Then fix any compact set $B \subset \Phi$, and choose functions $f_n \in \mathcal{C}_{00}^+(\Phi)$ with $f_n \downarrow \mathbf{1}_B$. Then $\int f_n(x)\mu(dx) \downarrow \mu(B)$ for every $\mu \in \mathcal{M}(\Phi)$ and so the mapping p_B is $\mathcal{F}$–measurable. Apply Dynkin–Doob monotone lemma to all Borel subsets of B to see that p_B is $\mathcal{F}$–measurable for any measurable relatively compact subset of Φ. Hence $\mathcal{G} \subset \mathcal{F}$. □

It is clear from the definition that if X is a point process on $(\Psi, \mathcal{B})$ then

(1) The map $\omega \mapsto X(B, \omega)$ is a $\overline{Z}_+$–valued random variable
(2) For fixed $\omega \in \Omega$, the map $B \mapsto X(B, \omega)$ defined on $\mathcal{B}$ is a $\overline{Z}_+$–valued measure.

If in addition if $X(B)$ is Poisson distributed for each $B \in \mathcal{B}$, then we say that X is a *Poisson point process.*

A Poisson process $\{N_t : t \geq 0\}$ with parameter c, introduced in section 2, can be thought of as a Poisson point process on $([0,\infty), \mathcal{B}([0,\infty)))$ with intensity measure $c\lambda$, where λ is Lebesgue's measure on $[0,\infty)$. In this case $N([0,t),\omega) = N_t(\omega)$.

The following pair of results establish the existence of Poisson point–processes with given intensity measures and some of their distributional properties.

Proposition 3.2. *Let $(\Psi, \mathcal{B})$ a σ–space and $\rho \in \Sigma_{\mathcal{M}}$. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Poisson point process X with intensity measure ρ such that for each finite measurable partition $B_1, \ldots, B_k$ of Ψ, the random variables $X(B_1), \ldots, X(B_k)$ are independent and Poisson distributed with means $\rho(B_1), \ldots, \rho(B_k)$ respectively.*

Proof. Assume first that ρ is finite. If $\rho = 0$ then we take $N = 0$. If $\rho(\Psi) > 0$ then, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where a sequence $\{Z_n : n \in \mathbb{N}\}$ of independent identically distributed random variables on Ψ with distribution $\rho/\rho(\Psi)$ and a Poisson random variable N with mean $\rho(\Psi)$ and independent of (Z_n) can be conceived. We define the point processes

$$\begin{aligned} S_0(B,\omega) &= 0 \\ S_n(B,\omega) &= \sum_{j=1}^{n} \mathbf{1}_{B_j}(Z_j(\omega)) \end{aligned}$$

We use the variable N to randomize the index n and define

$$X(B,\omega) = S_{N(\omega)}(B,\omega)$$

Let $B_1, \ldots, B_k$ be a finite measurable partition of Ψ, $n_1, \ldots, n_k \in \mathbb{Z}_+$. Let $n_1 + \ldots + n_k = n$. Then

$$\begin{aligned}
&\mathbb{P}\left[X(B_1) = n_1, \ldots, X(B_k) = n_k\right] \\
&\qquad = \mathbb{P}\left[X(B_1) = n_1, \ldots, X(B_k)|X(\Psi) = n\right]\mathbb{P}[X(\Psi) = n] \\
&\qquad\quad = \mathbb{P}\left[\sum_{j=1}^{n} \mathbf{1}_{B_j}(Z_j) = n_j, \ldots, \sum_{j=1}^{n} \mathbf{1}_{B_k}(Z_j) = n_k\right]\mathbb{P}[N = n]
\end{aligned}$$

Since $B_j's$ are disjoint and $Z_j's$ are independent we gave that

$$\begin{aligned}
\mathbb{P}\Big[X(B_1) &= n_1, \ldots, X(B_k) = n_k\Big] \\
&= \tfrac{n!}{(n_1!)\cdots(n_k!)}\left(\tfrac{\rho(B_1)}{\rho(\Psi)}\right)^{n_1}\cdots\left(\tfrac{\rho(B_k)}{\rho(\Psi)}\right)^{n_k} e^{-\rho(\Psi)}\tfrac{\rho^n(\Psi)}{n!} \\
&= \prod_{j=1}^{k} e^{-\rho(B_j)}\tfrac{\rho^{n_j}(B_j)}{n_j!}
\end{aligned}$$

Summing over $n_1, \ldots, n_k$ except for n_j we get

$$\mathbb{P}[X(B_j) = n_j] = e^{-\rho(B_j)}\tfrac{\rho^{n_j}(B_j)}{n_j!}$$

This finishes the proof for the case $\rho < \infty$.

Assume next that $\rho(\Psi) = \infty$. Since $(\Psi, \mathcal{B})$ is a σ–space, there are disjoint sets $\Psi_1, \Psi_2, \ldots, \in \mathcal{B}$ such that $\Psi = \bigcup_{k=1}^{\infty}\Psi_k$ and $\rho(\Psi_k) < \infty$ for all k. Define the finite measures $\rho_k(B) = \rho(\Psi_k \bigcap B)$. Using the first part of the proof, we can construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of independent Poisson point process $(X_k : k \in \mathbb{N})$ with intensity measures ρ_k. We define the point process

$$X(B, \omega) = \sum_{k=1}^{\infty} X_k(B, \omega)$$

N is a Poisson point process with intensity measure ρ. Since

$$\mathbb{E}[X(B)] = \sum_{k=1}^{\infty}\mathbb{E}[X_k(B)] = \sum_{k=1}^{\infty}\rho_k(B) = \rho(B).$$

Since the sum of independent Poisson random variables is again Poisson, then $\omega \mapsto X(B, \omega)$ is Poisson distributed if $\rho(B) < \infty$. If $\rho(B) = \infty$ then

$$\sum_{k=1}^{\infty}\mathbb{P}[X_k(B) \geq 1] = \sum_{k=1}^{\infty}(1 - e^{-\rho(B)}) \geq \sum_{k=1}^{\infty}\tfrac{\rho(B_k)\wedge a}{2} = \infty$$

since for some $a \in [1,2]$, $1 - e^{-r} \geq \frac{r}{2} \wedge (1 - e^{-a})$. Thus, by Borel–Cantelli's lemma

$$\mathbb{P}\left[\limsup_{k\to\infty}[X_k(B) > 1]\right] = 1$$

This shows that $X(B,\omega) = \infty$ a.s for every ω if $\rho(B) = \infty$. □

Proposition 3.3. *Let be $(\Psi, \mathcal{B}, \rho)$ be a measure space with $\rho(\Psi) < \infty$ and let X be a Poisson point process in $(\Psi, \mathcal{B})$ with intensity measure ρ. Let be $\varphi : \Psi \mapsto \mathbb{R}^d$ be a measurable function and defined*

$$Y(\omega) = \int_\Psi \varphi(\psi) X(d\psi, \omega)$$

Then the following are true:

(i) Y is a random variable on $\mathbb{R}^d$ with compound Poisson distribution satisfying

$$\mathbb{E}\left[e^{i\langle z,Y\rangle}\right] = \exp\left[\int_{\mathbb{R}^d}\left(e^{i\langle x,x\rangle} - 1\right)(\rho\varphi^{-1})(dx)\right] \tag{56}$$

(ii) If $\varphi \in L_2(\rho)$ then $Y \in L_2(\mathbb{P})$ and

$$\begin{aligned} \mathbb{E}[Y] &= \int_\Psi \varphi(\psi)\rho(d\psi) &(57)\\ \mathbb{E}\left[|Y - \mathbb{E}[Y]|^2\right] &= \int_\Psi |\varphi(\psi)|^2\rho(\psi) &(58)\end{aligned}$$

(iii) Suppose that $B_1, \ldots, B_m$ are disjoint sets in $\mathcal{B}$ and let

$$Y_k(\omega) = \int_{B_k} \varphi(\psi) X(d\psi, \omega)$$

Then $Y_1, \ldots, Y_m$ are independent.

Proof. (i) From the assumption that $\rho(\Psi) < \infty$, it follows that $X(\Psi, \omega) < \infty$ almost surely in ω. Hence, the measure $X(d\psi, \omega)$ is supported on a finite number of points and $Y(\omega)$ is almost surely finite. For each $p \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, consider the boxes $C_p^n = \{x = (x_1, \ldots, x_d) : 2^{-n}(p_j - 1) < x_j \leq 2^{-n}p_j\}$. For each $n \in \mathbb{N}$ the family $\{C_p^n : p \in \mathbb{Z}^d\}$ covers the space $\mathbb{R}^d$. Choose $x_p^n \in C_p^n$ and define the sequence of functions $\varphi_n : \Psi \mapsto \mathbb{R}^d$ by setting $\varphi_n(\psi) = x_p^n$ if $\psi \in \varphi^{-1}(C_p^n)$. Then

$$\sup_{\psi\in\Psi} |\varphi_n(\psi) - \varphi(\psi)| \leq 2^{-n}\sqrt{d}$$

which implies that for almost surely all ω

$$|Y_n(\omega) - Y(\omega)| \leq 2^{-n}\sqrt{d}X(\Psi, \omega) \to 0 \quad \text{as} \quad n \to \infty$$

It follows that $Y_n(\omega)$ is measurable. Since

$$Y_n(\omega) = \sum_{p\in\mathbb{Z}^d} x_p^n X(\varphi^{-1}(C_p^n),\omega),$$

and $X(\varphi^{-1}(C_p^n),\omega)$ is Poisson distributed with parameter $\rho(\varphi^{-1}(C_p^n))$ then

$$\begin{aligned}
\mathbb{E}\left[e^{i\langle x,Y_n\rangle}\right] &= \prod_{p\in\mathbb{Z}^d}\mathbb{E}\left[e^{i\langle z,x_p^n X(\varphi^{-1}(C_p^n))\rangle}\right]\\
&= \prod_{p\in\mathbb{Z}^d}\exp\left[(e^{i\langle z,x_p^n\rangle}-1)\rho(\varphi^{-1}(C_p^n))\right]\\
&= \exp\left[\int_\Psi (e^{i\langle z,\varphi_n(\psi)\rangle}-1)\rho(d\psi)\right]
\end{aligned}$$

Therefore, dominated convergence shows that Y is a compound Poisson random variable in $\mathbb{R}^d$.

(ii) The assumptions that $\rho(\Psi)<\infty$ and $\int_\Psi|\varphi(\psi)|^2\rho(d\psi)$ allow us to use dominated convergence to interchange the symbols of differentiation and integration to get

$$\begin{aligned}
\tfrac{1}{i}\tfrac{\partial}{\partial z_j}\int_\Psi (e^{i\langle z,\varphi(\psi)\rangle}-1)\rho(d\psi) &= \int_\Psi \varphi_j(\psi)e^{i\langle z,\varphi(\psi)\rangle}\rho(d\psi)\\
\left(\tfrac{1}{i}\tfrac{\partial}{\partial z_j}\right)^2\int_\Psi (e^{i\langle z,\varphi(\psi)\rangle}-1)\rho(d\psi) &= \int_\Psi \varphi^2(\psi)e^{i\langle z,\varphi(\psi)\rangle}\rho(d\psi)
\end{aligned}$$

Then (ii) follows from theorem 1.16 (ix).

(iii) Using the sequence $\varphi_n(\psi)$ introduced before, let

$$Y_{n,k}(\omega) = \int_{B_k}\varphi_n(\psi)X(d\psi,\omega) = \sum_{p\in\mathbb{Z}^d} x_p^n X(B_k\cap C_p^n,\omega)$$

for $k=1,\dots,m$. From proposition 3.2 we get that $X(B_k\cap C_p^n,\omega)$ with $1\le k\le m$ and $p\in\mathbb{Z}^d$ are independent. Therefore $Y_{n,1},\dots,Y_{n,m}$ are independent and since $Y_{n,k}(\omega)\to Y_k(\omega)$ as $n\to\infty$, it follows that $Y_1,\dots,Y_m$ are independent. $\square$

We need to specialize some of the basic concepts we have introduced so far. Let us fixed two locally compact Polish spaces S and K equipped with their Borel σ–algebras $\mathcal{S}$ and $\mathcal{K}$ respectively. Let X be a point process on $S\times K$. Then,

(i) X is a K*–marked point process* on S if

$$X(\{s\}\times K,\cdot)\le 1 \qquad \text{for all } s\in S$$

(ii) X has independent increments in S if the point processes on $(K, \mathcal{K})$ defined by $X(B_1 \times \cdot), \ldots, X(B_n \times \cdot)$ are independent for any disjoint sets $B_1, \ldots, B_n \in \widehat{\mathcal{S}}$

Theorem 3.4. *(Erlang, Lévy) Let X be a K–marked point process on S such that $X(\{s\} \times K, \omega) = 0$ almost surely in ω for all $S \in S$. Then X is Poisson on $S \times K$ iff it has independent increments, in which case $\mathbb{E}X$ is locally finite, in the sense that any point $x \in S \times K$ has a neighborhood G_s with $\mathbb{E}X(G_s) < \infty$.*

The proof will be based on properties of *dissecting systems.* By a dissecting system of a Topological space S we mean an array $\{D_{nj}\}$ of Borel sets such that for any $n \in \mathbb{N}$, $\mathcal{D}_n = \{D_{nj} : j = 1, \ldots, m_n\}$ is a partition of S, $\mathcal{D}_{n+1}$ is a refinement of $\mathcal{D}_n$, and the following property holds: For any compact $K \subset S$ with any open cover $\{G_i : i \in I\}$, there exists an integer N such that for $n \geq N$, $D_{nj} \bigcap K$ is contained in some G_i.

For locally compact Polish spaces such a dissecting system exists. To construct one we start by considering a countable basis $B_1, \ldots$ and then, we generate $\mathcal{D}_n$ by considering finite partitions of $B_1, \ldots, B_n$ of diameter not larger that 2^{-n}. It is not difficult to verify the dissecting property by using the Lebesgue number of any given open cover of any given compact set K.

Lemma 3.5. *(Dissection properties) Let X be a simple point process on S with $X(\{s\}, \omega) = 0$ a.s. in ω for all $s \in S$. Let $B \in \widehat{S}$ and $\{D_{nj}\}$ a dissecting system and set $B_{nj} = B \bigcap D_{nj}$. Then,*

(i) $\max_j X(B_{nj}, \omega) \vee 1 \to 1$ *a.s in* ω *as* $n \to \infty$.

(ii) $\max_j \mathbb{P}\left[X(B_{nj}) > 0\right] \to 0$ *as* $n \to \infty$

Proof. (i) If $\mu \in \Sigma_{\mathcal{S}}$ is simple, then for each $s \in \bar{B}$ there is an open set $G_s \ni s$ with $\mu(G_s) \leq 1$. By the dissecting property we may choose n large enough so that B_{nj} is contained in some G_s. Then $\max_j \mu(B_{nj}) \leq 1$.

(ii) Let us fix $\varepsilon > 0$. For each $s \in \bar{B}$ we have $\mathbb{P}[X(\{s\}) > 0] = 0$. Let G_n^s be a decreasing sequence of relatively compact open sets containing S such that $\bigcap G_n^s = \{s\}$. Then $X(\{s\}, \omega) = \lim_n X(G_n^s, \omega)$ leads to

$$0 = \mathbb{P}\left[X(\{s\}) > 0\right] = \lim_n \mathbb{P}\left[X(G_n^s) > 0\right]$$

Thus, for some integer N large enough, $\mathbb{P}\left[X(G_N^s) > 0\right] < \varepsilon$. By the dissecting property we may choose N even larger so that B_{nj} is contained in some G_s. This means that $\max_j \mathbb{P}[X(B_{nj}) > 0] < \varepsilon$ for $n \geq N$. This finishes up the proof of the present lemma. $\square$

Proof of theorem 3.4 Let us fix any relatively compact Borel set $B \subset S \times K$, then $\eta(A, \omega) = X(B \cap (A \times K), \omega)$ is a simple point process on S with independent increments and $\eta(\{s\}, \omega) = 0$ a.s in ω for all s. We introduce a dissecting system D_{nj} for S. Lemma 3.5 implies that $\eta(D_{nj}, \cdot)$ forms a null array with $\max_j (\eta(D_{nj}, \omega) \vee 1) \to 1$ a.s. Theorem 1.18 implies that $X(B, \cdot) = \eta(S, \cdot) = \sum_j \eta(D_{nj}, \cdot)$ is a Poisson random variable, and in particular $\mathbb{E}X(B) < \infty$. The rest follows from noticing that any locally compact Polish space is σ–compact. $\square$

We conclude this section by introducing the following notation:

$$\begin{aligned} D_{a,b} = D(a,b] \quad &= \{x \in \mathbb{R}^d : a < |x| \leq b\} \qquad \text{for } 0 \leq a < b < \infty \\ D_{a,\infty} = D(a,\infty) \quad &= \{x \in \mathbb{R}^d : a < |x| \leq \infty\} \qquad \text{for } 0 \leq a < \infty \end{aligned}$$

Thus, $D_{0,\infty} = \mathbb{R}^d \setminus \{0\}$. Further

$$H = (0,\infty) \times (\mathbb{R}^d \setminus \{0\}) = (0,\infty) \times D_{0,\infty}.$$

A point $h \in H$ is denoted by $h = (s,t)$. The Borel σ–algebra of H is denoted by $\mathcal{B}(H)$.

3.2. **Analytic formulation of the Itô–Lévy representation.** The Itô–Lévy representation is constituted by the following couple of theorems, the first deals with a general additive process, and the second with an additive process satisfying

$$\int_{|x|<1} |x|\nu_t(dx) < \infty \quad \text{for every } t \geq 0. \tag{59}$$

Theorem 3.6. *Let $\{X_t : t \geq 0\}$ be an additive process on $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with system of generating triplets $\{(A_t, \nu_t, \gamma_t)\}$ and define the measure $\tilde{\nu}$ on H by $\tilde{\nu}((0,t] \times B) = \nu_t(B)$ for $B \in \mathcal{B}(\mathbb{R}^d)$. With Ω_0 as in Definition 1.12 of an additive process, define for all $B \in \mathcal{B}(H)$*

$$J(B,\omega) = \begin{cases} \#\{s : (s, X_s(\omega) - X_{s-}(\omega)) \in B\} & \text{for } \omega \in \Omega_0 \\ 0 & \text{for } \omega \notin \Omega_0 \end{cases} \tag{60}$$

Then the following hold:

(i) $\{J(B) : B \in \mathcal{B}(H)\}$ is a Poisson point process on H with intensity measure $\tilde{\nu}$.

(ii) There is $\Omega_1 \in \mathcal{F}$ with $\mathbb{P}[\Omega_1] = 1$ such that for any $\omega \in \Omega_1$,

$$\begin{aligned} X_t^1(\omega) = &\lim_{\varepsilon \downarrow 0} \int_{(0,t]\times D(\varepsilon,1]} \{x\, J(d(s,x),\omega) - x\, \tilde{\nu}(d(s,x))\} \\ &+ \int_{(0,t]\times D(1,\infty)} x\, J(d(s,x),\omega) \end{aligned} \tag{61}$$

is defined for all $t \geq 0$ and the convergence is uniformly in t on any bounded interval. The process $\{X_t^1\}$ is an additive process on $\mathbb{R}^d$ with $\{(0,\nu_t,0)\}$ as the system of generating triplets.

(iii) Define

$$X_t^{(2)}(\omega) = X_t(\omega) - X_t^{(1)}(\omega) \quad \textit{for}\, \omega \in \Omega_1. \tag{62}$$

There is $\Omega_2 \in \mathcal{F}$ with $\mathbb{P}[\Omega_2] = 1$ such that for any $\omega \in \Omega_2$, $X_t^{(2)}(\omega)$ is continuous in t. The process $\{X_t^{(2)}\}$ is an additive process on $\mathbb{R}^d$ with $\{(A_t,0,\gamma_t)\}$ as the system of generating triplets.

(iv) The two process $\{X_t^{(1)}\}$ and $\{X_t^{(2)}\}$ are independent.

We will see that $\int_{(0,t]\times D(\varepsilon,1]}\{x\, J(d(s,x),\omega) - x\, \tilde{\nu}(d(s,x))\}$ has mean 0. Its limit as $\varepsilon \downarrow 0$ is called the *compensated sum of jumps*. Without the subtraction, the sum of jumps $\int_{(0,t]\times D(\varepsilon,1]} x\, J(d(s,x),\omega)$ may not converge as $\varepsilon \downarrow 0$.

Theorem 3.7. *Suppose that the additive process $\{X_t\}$ of theorem 3.6 satisfies in addition (59). Let $\gamma_0(t)$ be the drift of X_t. Then, there is $\Omega_3 \in \mathcal{F}$ with $\mathbb{P}[\Omega_3] = 1$ such that for any $\omega \in \Omega_3$,*

$$X_t^3(\omega) = \int_{(0,t]\times D(0,\infty)} x\, J(d(s,x),\omega) \tag{63}$$

is defined for all $t \geq 0$. The process $\{X_t^{(3)}\}$ is an additive process on $\mathbb{R}^d$ with $(0,\nu,0)_0$ as the system of generating triplets, i.e,

$$\mathbb{E}\left[e^{i\,\langle z, X_t^{(3)}\rangle}\right] = \exp\left[\int_{\mathbb{R}^d} \left(e^{i\,\langle z,x\rangle} - 1\right)\, \nu_t(dx)\right]. \tag{64}$$

Define

$$X_t^4(\omega) = X_t(\omega) - X_t^{(3)}(\omega) \quad \textit{for}\, \omega \in \Omega_3. \tag{65}$$

Then, for any $\omega \in \Omega_2 \bigcap \Omega_3$, $X_t^{(4)}(\omega)$ is continuous in t, and $\{X_t^{(4)}\}$ is an additive process on $\mathbb{R}^d$ with $(A_t,0,\gamma_0(t))_0$ as the system of generating triplets, i.e.,

$$\mathbb{E}\left[e^{i\,\langle z, X_t^{(4)}\rangle}\right] = \exp\left[\tfrac{1}{2}\langle z,x\rangle + i\,\langle \gamma_0(t), z\rangle\right]. \tag{66}$$

The two processes $\{X_t^{(3)}\}$ *and* $\{X_t^{(4)}\}$ *are independent.*

In theorem 3.6, we call $\{X_t^{(1)}\}$ and $\{X_t^{(2)}\}$ the *jump part* and the *continuous* part of $\{X_t\}$ respectively. In the same way, in theorem 3.7, $\{X_t^{(3)}\}$ and $\{X_t^{(4)}\}$ are also called the jump part ant the continuous part of $\{X_t\}$ respectively. The difference, however, is that these decompositions depend on the choice of representation of the system of generating triplets. In the former theorem, we use the canonical representation system (A, ν, γ); whereas in the latter, the "$c = 0$"–representation system $(A, \nu, \gamma_0)_0$ is used.

The Proof of the Itô–Lévy decomposition will be presenting is taken from [7], and it is divided in a series of lemmas. The idea is to construct an additive process $\{Y_t\}$ on some probability space such that $\{Y_t\} \stackrel{d}{=} \{X_t\}$. Then use the fact that both $\{Y_t\}$ and $\{X_t\}$ induce the same probability measure on the cadlad space $\mathbf{D} = \mathcal{D}([0,\infty), \mathbb{R}^d$ with σ–algebra $\mathcal{F}_{\mathbf{D}}$ generated by the Borel cylinder sets and that all the relevant terms are $\mathcal{F}_{\mathbf{D}}$–measurable.

Assume that we are given an additive process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ defined in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(A_t, \nu_t, \gamma(t))$ be the system of generating triplets. By Remark 2.1 there is a unique σ–finite Radon measure $\tilde{\nu}$ on H such that $\tilde{\nu}((0,t] \times B) = \nu_t(B)$ for all $t > 0$ and $B \in \mathcal{B}\left(D(0,\infty)\right)$. By Proposition 3.2 there exists another probability space $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ and a Poisson point process $\{N(B) : B \in \mathcal{B}(H)\}$ on H whose intensity measure is $\tilde{\nu}$.

Lemma 3.8. *There is* $\Omega^0_1 \in \mathcal{F}^o$ *with* $\mathbb{P}[\Omega^0_1] = 1$ *such that for any* $\omega \in \Omega^0_1$, *the following hold*

(i) for any $\varepsilon > 0$ *and* $t > 0$, *the measure* $N\left(\cdot \cap (0,t] \times D_{\varepsilon,\infty}, \omega\right)$ *is supported on a finite number of points, each of which has* $N(\cdot, \omega)$*–measure* 1*;*

(ii) for any $s > 0$, $N\left(\{s\} \times D_{0,\infty}, \omega\right) \leq 1$.

Proof. Write $H_{t,\varepsilon} = (0,t] \times D_{\varepsilon,\infty}$. Then from $\mathbb{E}^0[N(H_{t,\varepsilon})] = \tilde{\nu}(H_{t,\varepsilon}) < \infty$, we have $\mathbb{P}^0[N(H_{t,\varepsilon}) < \infty] = 1$. Choose $t_k \uparrow \infty$ and $\varepsilon_k \downarrow 0$ and set

$$\Omega' = \bigcap_{k=1}^{\infty} \left[N(H_{t_k,\varepsilon_k}) < \infty\right].$$

Then $\mathbb{P}^0[\Omega'] = 1$ and for any $\omega \in \Omega'$, $t > 0$ and $\varepsilon > 0$, we have that the measure $N\left(\cdot \bigcap H_{t,\varepsilon}, \omega\right)$ is supported in a finite number of points. Let N^* be the Poisson point process on H constructed as in the proof of Proposition 3.2 with $\Psi_1 = H_{t_1,\varepsilon_1}$ and for $k > 1$

$$\Psi_k = H_{t_k,\varepsilon_k} \setminus H_{t_{k-1},\varepsilon_{k-1}}$$

For Ψ_k fixed, let Z_n be the sequence of independent random variables as in step 1 of the proof of Proposition 3.2. Let us denote by $Z_{n,1}$ the component in $(0, t_k]$ of Z_n. If $n \neq m$, then

$$\mathbb{P}^0[Z_{n,1} = Z_{m,1}] = \int_{(0,t_k]} \mathbb{P}^0[Z_{n,1} = s]\mathbb{P}^0[Z_{m,1} \in ds] = 0$$

by Fubini's theorem and by $\mathbb{P}^0[Z_{n,1} = s] \leq \tilde{\nu}(\{s\} \times D_{\varepsilon_k,\infty})/\tilde{\nu}(\Psi_k) = 0$ as in condition (54) given in Remark 2.1. Since

$$\begin{aligned}
&[\exists s > 0 \text{ such that } N^*(\Psi_k \cap (\{s\} \times D_{0,\infty})) \geq 2] \\
&\quad \subset [\exists n, \exists m, \exists s > 0, \text{ such that } n \neq m \text{ and } Z_n, Z_m \in \{s\} \times D_{0,\infty}] \\
&\quad \subset [\exists n, \exists m, \text{ such that } n \neq m \text{ and } Z_{n,1} = Z_{m,1}]
\end{aligned}$$

and the last event has probability 0, it follows that $N^*(\Psi_k \cap (\{s\} \times D_{0,\infty})) \leq 1$ for all $s > 0$ a.s. Using the same argument, we can show that for $H_k = \bigcup_{l=1}^k \Psi_l$ that $N^*(H_k \cap (\{s\} \times D_{0,\infty})) \leq 1$ for all $s > 0$ a.s. Letting $k \to \infty$, we see that $N^*(\{s\} \times D_{0,\infty}) \leq 1$ for all $s > 0$ a.s.

For fixed $\varepsilon > 0$, let us define $Y_t = N(H_{t,\varepsilon})$ for $t > 0$ and $Y_0 = 0$. Then $\{Y_t : t \geq 0\}$ is an additive process. The stochastic continuity follows from (54) in Remark 2.1. Paths of $\{Y_t\}$ are non–decreasing right–continuous step functions with jump sizes being positive integers when we take the restriction to Ω'. Let $U_k = \inf\{t : Y_t \geq k\}$, the cadlag property of paths implies that $U_k = \inf\{t \in \mathbb{Q} : Y_t \geq k\}$. Note that the assertion $N(\{s\} \times D_{0,\infty}) \leq 1$ for all $s > 0$ a.s. is equivalent to the assertion

$$\mathbb{P}^0[U_1 < U_2 < \cdots < U_k < \infty] = \mathbb{P}^0[U_k < \infty] \quad \text{for each } k$$

These probabilities are identical with those for $\{Y_t^*\}$ similarly constructed from N^*. But by Lemma 3.1 we have $N \stackrel{d}{=} N^*$, which implies that $Y_t \stackrel{d}{=} Y_t^*$ and $U_k \stackrel{d}{=} U_k^*$. □

Going back to the Poisson point process N on H with intensity measure $\tilde{\nu}$, we have the following result.

Lemma 3.9. *For any sequence (ε_n) in $(0, 1]$ with $\varepsilon \downarrow 0$, there is $\Omega_2^0 \in \mathcal{F}^0$ with $\mathbb{P}[\Omega_2^0] = 1$ such that for any $\omega \in \Omega_2^0$,*

$$\int_{(0,t]\times D(\varepsilon_n,1]} \{x\, N(d(s, x), \omega) - x\, \tilde{\nu}(d(s, x))\} \tag{67}$$

converges to an element of $D([0, \infty), \mathbb{R}^d)$ uniformly on any bounded time interval as $n \to \infty$.

Proof. Let $\varepsilon_0 = 1$ and for $n \geq 1$

$$\begin{aligned} Z_n(t) &= \int_{(0,t]\times D(\varepsilon_n,\varepsilon_{n-1}]} \{x\, N(d(s,x),\omega) - x\, \tilde{\nu}(d(s,x))\} \\ S_n(t) &= \sum_{j=1}^{n} Z_j(t). \end{aligned}$$

Notice that $S_n(t)$ equals the integral in (67). By Proposition 3.3 we have that $\mathbb{E}[S_n(t)] = 0$ and

$$\mathbb{E}[|S_m(t) - S_n(t)|^2] = \int_{D(\varepsilon_m,\varepsilon_n]} |x|^2 \nu_t(dx) \quad \text{for } m > n.$$

By Lemma 3.8 it follows that $Z_n(t)$ is right continuous with left limits a.s. Fix t and let $r_0, r_1, \ldots,$ an enumeration of $([0,t) \cap \mathbb{Q}) \cup \{t\}$ with $r_0 = 0$ and $r_1 = t$. Then

$$\begin{aligned} &\mathbb{P}\left[\sup_{s\in[0,t]} |S_m(s) - S_n(s)| > \varepsilon\right] \\ &\qquad\qquad = \lim_{q\to\infty} \mathbb{P}\left[\max_{0\leq j\leq q} |S_m(r_j) - S_n(r_j)| > \varepsilon\right] \end{aligned} \tag{68}$$

For fixedd q, let $0 = s_0 < s_1 < \ldots < s_q = t$ be the ordering of $\{r_0, \ldots, r_q\}$. Then

$$S_m(t) - S_n(t) = \sum_{j=1}^{q} \int_{(s_{j-1},s_j]\times D(\varepsilon_m,\varepsilon_n]} \{x\, N(d(s,x)) - x\, \tilde{\nu} d(s,x))\}.$$

By Proposition 3.3 the right–hand side is a sum of independent random variables. Kolmogorov's inequality, lemma 1.22, we have that the right–hand side of equation (68) is bounded by

$$\frac{1}{\varepsilon^2} \int_{D(\varepsilon_m,\varepsilon_n]} |x|^2 \nu_t(dx),$$

which tends to 0 as $m, n \to \infty$. Finally, Lemma 1.21 we conclude that $\{S_n(t)\}$ converges to a limit $\{S(t)\}$ uniformly on any bounded time interval a.s. and that the limit is an element of $D([0,\infty), \mathbb{R}^d)$. $\square$ In the next lemma we improve the past result so that the uniform convergence does not depend on the choice of the sequence $\varepsilon \downarrow 0$.

Lemma 3.10. *(i) Let*

$$S_\varepsilon(t,\omega) = \int_{(0,t]\times D(\varepsilon,1]} \{x\, N(d(s,x),\omega) - x\, \tilde{\nu}(d(s,x))\}. \tag{69}$$

Then there exists $\Omega_3^0 \in \mathcal{F}^0$ *with* $\mathbb{P}[\Omega_3^0] = 1$ *such that for any* $\omega \in \Omega_3^0$, $S_\varepsilon(t,\omega)$ *converges uniformly on any bounded time interval as* $\varepsilon \downarrow 0$.

(ii) Define

$$Y_t^{(1)}(\omega) = \lim_{\varepsilon\downarrow 0} S_\varepsilon(t,\omega) + \int_{(0,t]\times D(1,\infty)} x\, N(d(s,x),\omega). \tag{70}$$

for $\omega \in \Omega_3^0$. *Then* $\left\{Y_t^{(1)}\right\}$ *is an additive process with generating triplets* $(0,\nu_t,0)$.

Consider the space $D([0,\infty),\mathbb{R}^d)$ with the norm of uniform convergence in compacta

$$\|\xi\| = \sum_{n=1}^{\infty} 2^{-n}\left(1 \wedge \sup_{t\geq 0} |\xi(t\wedge n)|\right)$$

Note that

$$\limsup_{\varepsilon,\varepsilon'\to 0} \|S_\varepsilon(\cdot,\omega) - S_{\varepsilon'}(\cdot,\omega)\| = \lim_{n\to\infty} \sup_{\varepsilon,\varepsilon'\in(0,1/n)} \|S_\varepsilon(\cdot,\omega) - S_{\varepsilon'}(\cdot,\omega)\|$$

Since for every $\omega \in \Omega_1^0$, with Ω_1^0 is as in Lemma 3.8, and $a > 0$ the support of $N(\cdot\cap(0,t]\times D_{a,\infty},\omega)$ consists of finite number of points, then $S_\varepsilon(\cdot,\omega)$ can be approximated in $\|\|$–norm by $S_{\varepsilon'}(\cdot,\omega)$ with $\varepsilon' \in \mathbb{Q}$. Thus,

$$\sup_{\varepsilon,\varepsilon'\in(0,1/n)} \|S_\varepsilon(\cdot,\omega) - S_{\varepsilon'}(\cdot,\omega)\| = \sup_{\varepsilon,\varepsilon'\in\mathbb{Q}\cap(0,1/n)} \|S_\varepsilon(\cdot,\omega) - S_{\varepsilon'}(\cdot,\omega)\|$$

For any n there is a finite set $\Lambda_n \subset \mathbb{Q}\cap(0,1/n)$ such that

$$\sup_{\varepsilon,\varepsilon'\in\mathbb{Q}\cap(0,1/n)} \|S_\varepsilon(\cdot,\omega) - S_{\varepsilon'}(\cdot,\omega)\| - \sup_{\varepsilon,\varepsilon'\in\Lambda_n} \|S_\varepsilon(\cdot,\omega) - S_{\varepsilon'}(\cdot,\omega)\| < \tfrac{1}{n}$$

Let $\Lambda = \bigcup_n \Lambda_n$. One can rearrange Λ in a decreasing sequence $\varepsilon_n \downarrow 0$. It follows that

$$\limsup_{\varepsilon,\varepsilon'\to 0} \|S_\varepsilon(\cdot,\omega) - S_{\varepsilon'}(\cdot,\omega)\| = \limsup_{j,k\to\infty} \|S_{\varepsilon_j}(\cdot,\omega) - S_{\varepsilon_k}(\cdot,\omega)\| \quad \text{a.s.}$$

Now we use lemma 3.9 and get the existence of $\Omega_3^0 \in \mathcal{F}^0$ with $\mathbb{P}[\Omega_3^0] = 1$ such that for any $\omega \in \Omega_3^0$

$$\limsup_{\varepsilon,\varepsilon'\to 0} \|S_\varepsilon(\cdot,\omega) - S_{\varepsilon'}(\cdot,\omega)\| = 0$$

This shows that $S_\varepsilon(\cdot,\omega)$ converges to a function in $D([0,\infty),\mathbb{R}^d)$ uniformly on any bounded time interval as $\varepsilon \downarrow 0$. Now consider $Y_t^{(1)}$

previously defined above. Corollary 1.17 and Proposition 3.3 imply that $\left\{Y_t^{(1)}\right\}$ has independent increments and that

$$\begin{aligned}\mathbb{E}\left[e^{i\,\langle z, Y_t^{(1)}\rangle}\right] &= \lim_{\varepsilon\downarrow 0}\mathbb{E}\left(\exp\left[i\langle z, S_\varepsilon(t,\omega)\rangle + \int_{(0,t]\times D(1,\infty)} x\, N(d(s,x),\omega)\right]\right)\\ &= \lim_{\varepsilon\downarrow 0}\exp\left[\int_{D(\varepsilon,\infty)}(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbf{1}_{D(\varepsilon,1]}(x))\nu_t(dx)\right]\\ &= \exp\left[\int_{D(0,\infty)}(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbf{1}_{D(0,1]}(x))\nu_t(dx)\right]\end{aligned}$$

This shows that $\left\{Y_t^{(1)}\right\}$ is an additive process with generating triplets $(0, \nu_t, 0)$. □

Let $\left\{Y_t^{(2)} : t \geq 0\right\}$ be an additive process on $\mathbb{R}^d$ having continuous paths with generating triplets $(A_t, 0, \gamma_t)$. This in fact will be a Gaussian process with mean γ_t and covariance matrix A_t, so its existence is easily guaranteed say by either Theorem 1.33 or Theorem 2.19. By enlarging the space $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ id necessary, we construct $\left\{Y_t^{(2)}\right\}$ on Ω^0 so that $\left\{Y_t^{(1)}\right\}$ and $\left\{Y_t^{(2)}\right\}$ are independent and then define

$$Y_t = Y_t^{(1)} + Y_t^{(2)} \tag{71}$$

Then $\{Y_t\}$ is an additive process with generating triplets (A_t, ν_t, γ_t).

Lemma 3.11. *There is $\Omega_4^0 \in \mathcal{F}^0$ with $\mathbb{P}^0[\Omega_4^0] = 1$ such that for any $\omega \in \Omega_4^0$ and $b \in \mathcal{B}(H)$,*

$$N(B,\omega) = \#\{s : (s, Y_s - Y_{s-}) \in B\}. \tag{72}$$

Proof. Since $Y_t^{(2)}$ is continuous, $Y_s - Y_{s-} = Y_s^{(1)} - Y_{s-}(1)$. Let

$$V_\varepsilon(t) = \int_{(0,t]\times D(\varepsilon,\infty)}\{x\, N(d(s,x)) - \mathbf{1}_{D(\varepsilon,1]}(x)x\,\tilde{\nu}(d(s,x))\}. \tag{73}$$

Let $\Omega_4^0 \in \mathcal{F}^0$ with $\mathbb{P}^0[\Omega_4^0] = 1$ be such that *(i)* and *(ii)* in Lemma 3.8 hold and $V_\varepsilon(t,\omega)$ converges to $Y_t^{(1)}(\omega)$ uniformly on any finite time interval as $\varepsilon \downarrow 0$. For $\omega \in \Omega_4^0$ we have

$$Y_s^{(1)}(\omega) - Y_{s-}(1)(\omega) = \lim_{\varepsilon\downarrow 0}\left(V_\varepsilon(s,\omega) - V_\varepsilon(s-,\omega)\right).$$

If $N(\{(s,x)\},\omega) = 1$, then $N(\{s\}\times D_{0,\infty},\omega) = 1$ and $V_\varepsilon(s,\omega) - V_\varepsilon(s-,\omega) = x$, hence $Y_s^{(1)}(\omega) - Y_{s-}(1)(\omega) = x$. On the other hand, if $N(\{(s,x)\},\omega) = 0$, then $Y_s^{(1)}(\omega) - Y_{s-}(1)(\omega) = 0$. This shows (72). □

Consider $\mathbf{D} = D([0,\infty),\mathbb{R}^d)$ as a subspace of $(\mathbb{R}^d)^{[0,\infty)}$ with the product σ–algebra, which we will denote by $\mathcal{F}_{\mathbf{D}}$. It is not difficult to check that the Borel σ–algebra $\mathcal{B}(\mathbf{D})$ and $\mathcal{F}_{\mathbf{D}}$ coincide with each other. For $\xi \in \mathbf{D}$, we write $x(t,\xi) = x_t(\xi) = \xi(t)$. Now, we need to note that even though the set of jumping times of ξ is countable, not always can we enumerate those points in increasing order. We will introduce however, an enumeration that will come at handy for the rest of the proof of the Itô–Lévy decomposition. For each $n = 1,\ldots$, the number of jumps of ξ such that $\xi(t) - \xi(t-) \in D\left(\frac{1}{n}, \frac{1}{n-1}\right]$ (replaced by $D(1,\infty)$ if $n = 1$) is finite in any bounded time interval (because otherwise ξ either does not have right limit or does not have left limit at some time t). We order these jumping times in increasing order $0 < t_{n,1}(\xi) < t_{n,2}(\xi) < \cdots$. If $\#\left\{t : \xi(t) - \xi(t-) \in D\left(\frac{1}{n}, \frac{1}{n-1}\right]\right\} = k < \infty$, then we set $t_{n,k+1}(\xi) = t_{n,k+2}(\xi) = \cdots = \infty$.

Lemma 3.12. *For any n and j, $t_{n,j}(\xi)$ is $\mathcal{F}_{\mathbf{D}}$–measurable.*

Proof. Let $\mathbb{Q}_t = ((0,t)\cap\mathbb{Q})\cup\{t\}$. The assertion $t_{1,1}(\xi) \le t$ is equivalent to the assertion that there exists $k \in \mathbb{N}$ such that for any $m \in \mathbb{N}$, there are $r, q \in \mathbb{Q}_t$ such that $r < s < r + \frac{1}{m}$ and $|\xi(s) - \xi(r)| > 1 + \frac{1}{k}$. Hence $t_{1,1}$ is $\mathcal{F}_{\mathbf{D}}$–measurable. The assertion $t_{1,2}(\xi) \le t$ is equivalent to the assertion that there exists $k \in \mathbb{N}$ such that for any $m \in \mathbb{N}$, there are $r, s \in \mathbb{Q}_t$ satisfying $t_{1,1}(\xi) < r < s < r + \frac{1}{m}$ and $|\xi(s) - \xi(r)| > 1 + \frac{1}{k}$. Hence $t_{1,2}$ is $\mathcal{F}_{\mathbf{D}}$–measurable. Similarly we can show the $\mathcal{F}_{\mathbf{D}}$–measurability of other $t_{n,j}$. □

Proof of Theorem 3.6: Let $\{X_t\}$ be an additive process on $\mathbb{R}^d$ defined on some probability space $(\Omega, \mathcal{F}\,\mathbb{P})$, with (A_t, ν_t, γ_t) as a system of generating triplets. Using this system of generating triplets, we can construct another additive process $\{Y_t\}$ on $\mathbb{R}^d$ defined in some other probability space $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$, which has the Itô–Lévy decomposition given in theorem 3.6. furthermore, we might assume that on Ω^0 lemmas 3.8–3.11 hold true.
By Theorem 2.19 processes X and Y have the same law, and by theorem 2.21 we might assume that the paths of $\{X_t\}$ and of $\{Y_t\}$ are cadlag. Define mappings $\psi : \Omega \mapsto \mathbf{D}$ and $\psi^0 : \Omega^0 \mapsto \mathbf{D}$ by

$$x_t(\psi(\omega)) = X_t(\omega) \tag{74}$$
$$x_t(\psi^0(\omega^0)) = Y_t(\omega^0) \tag{75}$$

The equality in law implies that

$$\mathbb{P}[\psi^{-1}(G)] = \mathbb{P}^0[(\psi^0)^{-1}(G)] \tag{76}$$

fon any $G \in \mathcal{F}_{\mathbf{D}}$. Let us define $\mathbb{P}^{\mathbf{D}}[G]$ by the value of (76). Then, under $\mathbb{P}^{\mathbf{D}}$, $\{x_t\}$ is an additive process identical in law to both $\{X_t\}$ and $\{Y_t\}$.

For $\xi \in \mathbf{D}$ and $B \in \mathcal{B}(H)$ let

$$\theta(B,\xi) = \#\{s > 0 : (s, x_s(\xi) - x_{s-}(\xi)) \in B\}$$

which plays the same role as N does for Y. Let us define the sets

$$G(k,j,B) = \{\xi : t_{k,j}(\xi) < \infty \text{ and } x(t_{k,j}(\xi),\xi) - x(t_{k,j}(\xi)-,\xi) \in B\}.$$

Since the jumping times of ξ are exhausted by $t_{k,j}(\xi)$, we have that

$$\theta(B,\xi) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \mathbf{1}_{G(k,j)}(\xi)$$

By Proposition 1.5 $x(t,\xi)$ is $(\mathcal{B}([0,\infty)) \times \mathcal{F}_{\mathbf{D}})$–measurable in (t,ξ). Then, it follows from Lemma 3.12 that $x(t_{k,j}(\xi),\xi)$ and $x(t_{k,j}(\xi)-,\xi)$ are $\mathcal{F}_{\mathbf{D}}$–measurable. Hence $G(k,j) \in \mathcal{F}_{\mathbf{D}}$, which in turn, implies the$\mathcal{F}_{\mathbf{D}}$– measurability of $\theta(B,\xi)$ in ξ. With $\theta(B,\omega)$ as in theorem 3.6 we have

$$\theta(B,\omega) = \theta(B,\psi(\omega)) \quad \text{for} \quad \omega \in \Omega$$

and by lemma 3.11

$$N(B,\omega) = \theta(B,\psi^0(\omega^0)) \quad \text{for} \quad \omega^0 \in \Omega^0$$

By uniqueness of random measures, 3.1, it follows that J, N and θ are identical in law. This proves part *(i)* of Theorem 3.6.

We use the sets $G(k,j,t,\varepsilon) = G(k,j,(0,t] \times D(\varepsilon,\infty))$ to define

$$\begin{aligned} u_\varepsilon(t,\xi) &= \int_{(0,t]\times D(\varepsilon,\infty)} \{x\,\theta(d(s,x),\xi) - x\,\mathbf{1}_{D(\varepsilon,1]}(x)\tilde{\nu}(d(s,x))\} \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}[x_{t_{k,j}(\xi)}(\xi) - x_{t_{k,j}(\xi)-}(\xi)]\mathbf{1}_{G(k,j,t,\varepsilon)}(\xi) - \int_{D(\varepsilon,1]} x\,\nu_t(dx). \end{aligned}$$

Only a finite number of summands are non–zero. Define

$$U_\varepsilon(t,\omega) = u_\varepsilon(t,\psi(\omega)) \quad \text{for} \quad \omega \in \Omega.$$

Then,

$$U_\varepsilon(t,\omega) = \int_{(0,t]\times D(\varepsilon,\infty)} \{x\,J(d(s,x),\omega) - x\,\mathbf{1}_{D(\varepsilon,1]}(x)\tilde{\nu}(d(s,x))\}$$

Also it is clear that

$$V_\varepsilon(t,\omega^0) = u_\varepsilon(t,\psi^0(\omega^0)) \quad \text{for} \quad \omega^0 \in \Omega^0$$

Let

$$\mathbf{D}_0 = \left\{ \xi : \limsup_{\varepsilon,\varepsilon'\downarrow 0} \|u_\varepsilon(\cdot,\xi) - u_{\varepsilon'}(\cdot,\xi)\| = 0 \right\}$$

Since the lim sup above coincides with the limit as $n \to \infty$ of the supremum over $\varepsilon, \varepsilon' \in \mathbb{Q} \cap (0, 1/n)$, we have that $\mathbf{D}_0 \in \mathcal{F}_{\mathbf{D}}$. Let us define the set

$$\Omega_1 = \{\omega : U_\varepsilon(\cdot,\omega) \text{ converges uniformly on } \text{bounded time intervals as } \varepsilon \to 0\}$$

Then by Lemma 3.10

$$\mathbb{P}[\Omega_1] = \mathbb{P}[\mathbf{D}_0] = \mathbb{P}[\Omega^0] = 1$$

Set $X_t^{(1)}(\omega) = 0$ for $\omega \notin \Omega_1$ and let

$$x_t(\xi) = \begin{cases} \lim_{\varepsilon\downarrow 0} u_\varepsilon(t,\xi) & \text{for} \quad \xi \in \mathbf{D}_0 \\ 0 & \text{for} \quad \xi \notin \mathbf{D}_0 \end{cases}$$

Then

$$\begin{aligned} X_t^{(1)}(\omega) &= x_t^{(1)}(\psi(\omega)) \\ Y_t^{(1)}(\omega^0) &= x_t^{(1)}(\psi(\omega^0)). \end{aligned}$$

It follows that $\left\{X_t^{(1)}\right\}$ and $\left\{x_t^{(1)}\right\}$ are additive process identical in law with $\left\{Y_t^{(1)}\right\}$. This shows part *(ii)* of Theorem 3.6. For the final step of the proof, set

$$\begin{aligned} x_t^{(2)}(\xi) &= x_t(\xi) - x_t^{(1)}(\xi) && \text{for} \quad \xi \in \mathbf{D} \\ X_t^{(2)}(\xi) &= x_t^{(2)}(\psi(\omega)) && \text{for} \quad \omega \in \Omega_1 \end{aligned}$$

then by (71), (74) and (75), we have that

$$Y_t^{(2)}(\omega^0) = x_t^{(2)}(\psi^0(\omega^0)) \quad \text{for} \quad \omega^0 \in \Omega^0$$

If t is a point of discontinuity of ξ then for any small ε

$$u_\varepsilon(t,\xi) - u_\varepsilon(t-,\xi) = x_t(\xi) - x_{t-}(\xi)$$

taking $\varepsilon \downarrow 0$ we get that

$$x_t^{(1)}(\xi) - x_{t-}^{(1)}(\xi) = x_t(\xi) - x_{t-}(\xi)$$

from which $x_t^{(2)}(\xi) = x_{t-}^{(2)}(\xi)$. If t is a point of continuity of ξ then $u_\varepsilon(t,\xi) = u_\varepsilon(t-,\xi)$ for all ε, and passing to the limit as $\varepsilon \downarrow 0$ we get $x_t^{(1)}(\xi) = x_{t-}^{(1)}(\xi)$, which shows again that $x_t^{(2)}(\xi) = x_{t-}^{(2)}(\xi)$. Therefore, $X_t^{(2)}(\omega)$ is continuous in t for any $\omega \in \Omega_1$. The equality in law, equation (76), implies that the processes $\left\{(X_t^{(1)}, X_t^{(2)})\right\}$, $\left\{(x_t^{(1)}, x_t^{(2)})\right\}$, and

$\left\{(Y^{(1)}_t, Y^{(2)}_t)\right\}$ are all identical in law. Since $Y^{(1)}$ and $Y^{(2)}$ are independent, then $X^{(1)}$ and $X^{(2)}$ are independent as well; the identity in law of $Y^{(2)}$ and $X^{(2)}$ implies that $X^{(2)}$ is an additive process. Then parts *(iii)* and it (iv) hold true and the proof of Theorem 3.6 is complete.

Proof of theorem 3.7 Assume then that $\int_{|x|\leq 1}|x|\,\nu_t(dx)<\infty$. For any Borel set C contained in $D_{\varepsilon,\infty}$, set

$$Y'(C,\omega)=\int_{(0,t]\times C}|x|J(d(s,x),\omega)$$

By Proposition 3.3 it follows that $Y'(C)$ has a compound Poisson distribution and that

$$\mathbb{E}[e^{-uY'(C)}]=\exp\left[\int_C(e^{-u|x|}-1)\nu_t(dx)\right]\quad\text{for}\quad u>0$$

Taking $C=D_{\varepsilon,\infty}$ and letting $\varepsilon\downarrow 0$ we get

$$\begin{aligned}\mathbb{E}&\left[\exp\left(-u\int_{(0,t]\times D(0,\infty)}|x|J(d(s,x))\right)\right]\\&=\exp\left[\int_{D(0,\infty)}(e^{-u|x|}-1)\nu_t(dx)\right]\quad\text{for}\quad u>0\end{aligned}$$

The right–hand side goes to 1 as $u\downarrow 0$, thus

$$\int_{(0,t]\times D(0,\infty)}|x|J(d(s,x),\omega)<\infty\quad\text{a.s. in}\quad\omega$$

Hence $X^{(3)}_t$ is well defined by equation (63) and finite a.s. It also follows that

$$X^{(3)}_t(\omega)=X^{(1)}_t(\omega)+\int_{D(0,1]}x\,\nu_t(dx)$$

The process $X^{(4)}$ defined by equation (65) satisfies

$$X^{(4)}_t(\omega)=X^{(2)}_2(\omega)-\int_{D(0,1]}x\,\nu_t(dx)$$

And Theorem 3.7 is easily obtained from Theorem 3.6. □

3.3. **Probabilistic formulation of the Itô–Lévy decomposition.** The following approach to the Itô–Lévy decomposition is purely probabilistic and does not rely upon the Lévy–Kintchine characterization of additive process.

Theorem 3.13. *Let X be a cadlag process in $\mathbb{R}^d$ with $X_0 = 0$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. X is an additive process iff there is $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}[\Omega_0] = 1$ such that for any $\omega \in \Omega_0$*

$$\begin{aligned} X_t(\omega) = m_t + G_t + \lim_{\varepsilon \downarrow 0} \int_{(0,t] \times D(\varepsilon,1]} & x\{(J(d(s,x),\omega) - \tilde{\nu}(d(s,x))\} \\ & + \int_{(0,t]\times D[1,\infty)} x\, J(d(s,x),\omega) \end{aligned} \tag{77}$$

for some continuous function m with $m_0 = 0$, some continuous centered Gaussian process G with independent increments and $G_0 = 0$, and some independent Poisson point process J on H with intensity measure $\tilde{\nu}$ such that

$$\int_{(0,t]\times D(0,1]} |x|^2 \tilde{\nu}(d(s,x)) < \infty \quad \textit{for} \quad t \leq 0. \tag{78}$$

Proof. (Olav Kallenberg) The following proof starts by analyzing the jump structure of X. We introduce the point measure J as in the statement of Theorem 3.6. First we show that $J((s,t] \times B, \omega)$ is measurable with respect the σ–algebra $\sigma(X_u : s < u \leq t)$. Consider partitions $s = t_{n,0} < t_{n,1} < \ldots < t_{n,n} = t$ so that $max_{1 \leq k \leq n}(t_{n,k} - t_{n,k-1}) \to 0$ as $n \to \infty$. For any continuous function f on $\mathbb{R}^d$ that vanishes in a neighborhood of 0 we have

$$\sum_{k=1}^{n} f(X_{t_{n,k}} - X_{t_{n,k-1}}) \to \int_{(s,t]\times D(0,\infty)} f(x)\, J(d(u,x))$$

which implies that measurability of the integrals on the right. By usual approximation methods we may conclude that $J((s,t] \times B)$ is $\sigma(X_u : s < u \leq t)$–measurable for any compact set $B \subset \mathbb{R}^d \setminus \{0\}$. Then by a monotone class argument we extend the measurability to all $J(A)$ with $A \in \mathcal{B}(H)$.

Since X has independent increments, the same property holds for J. Corollary 1.12 implies that $J(\{s\} \times D(0,\infty)) = 0$. Then by Theorem 3.4 it follows that J is a Poisson point process.

The following theorem is fundamental for the present proof.

Lemma 3.14. *(Orthogonality and independence) Let X and Y be continuous in probability cadlag processes in $\mathbb{R}^d$ with $X_0 = Y_0 = 0$ such that (X,Y) has independent increments. Also assume that Y is a step process and $\Delta X \cdot \Delta Y = 0$ a.s. Then X and Y are independent.*

Proof. Define J in terms of Y. We might assume that the jumps of Y are bounded by composing Y with say arctangent. Since J is Poisson, then Y has integrable variation on any finite time interval. We only need to show that $(X_{t_1}, \dots, X_{t_n})$ and $(Y_{t_1}, \dots, Y_{t_n})$ are independent for any $t_1 < \dots < t_n$, and for this it suffices to show that $X_t - X_s$ and $Y_t - Y_s$ are independent for any $0 \leq s < t$. Without loss of generality, we might take $s = 0$ and $t = 1$.
Fix $u, v \in \mathbb{R}^d$ and introduce the following locally martingales

$$M_t = \frac{e^{i\langle u, X_t\rangle}}{\mathbb{E}\left[e^{i\langle u, X_t\rangle}\right]} \quad N_t = \frac{e^{i\langle u, Y_t\rangle}}{\mathbb{E}\left[e^{i\langle u, Y_t\rangle}\right]} \quad t \geq 0$$

Since N has bounded jumps, then it has integrable variation on any finite time interval. For $n \in \mathbb{N}$ and using the martingale property and dominated convergence we get

$$\begin{aligned} \mathbb{E}[M_1 N_1 - 1] &= \mathbb{E}\left[\sum_{k=1}^{n}(M_{k/n} - M_{(k-1)/n})(N_{k/n} - N_{(k-1)/n})\right] \\ &\to \mathbb{E}\left[\sum_{s\leq 1} \Delta M_s \Delta N_s\right] = 0 \end{aligned}$$

Thus, $\mathbb{E}[M_1 N_1] = 1$, and so

$$\mathbb{E}\left[e^{i(\langle u, X_1\rangle + \langle v, Y_1\rangle)}\right] = \mathbb{E}\left[e^{i\langle u, X_1\rangle}\right] \mathbb{E}\left[e^{i\langle v, Y_1\rangle}\right]$$

Thus the independence of X and Y follows. $\square$

Notice that $J((0,t] \times D(\varepsilon, \infty)) < \infty$ a.s for all $t, \varepsilon > 0$ because X is cadlag. Since J is Poisson, it follows that $\mathbb{E}[J((0,t] \times D(\varepsilon, \infty))] < \infty$.
Let us define for each $\varepsilon > 0$

$$X^{\varepsilon}_t(\omega) = \int_{(0,t]\times D(\varepsilon,\infty)} x\, J(d(s,x), \omega) = \sum_{s\leq t} \Delta X_s \,\mathbf{1}_{[|\Delta X_s| > \varepsilon]}$$

By Lemma 3.14 it is clear that X^{ε} and $X - X^{\varepsilon}$ are independent. By proposition 3.3 we get for any $\varepsilon, t > 0$ and $u \in \mathbb{R}^d \setminus \{0\}$

$$\begin{aligned} 0 &< \left|\mathbb{E}\left[e^{i\langle u, X_t\rangle}\right]\right| \leq \left|\mathbb{E}\left[e^{i\langle u, X^{\varepsilon}_t\rangle}\right]\right| \\ &= \left|\mathbb{E}\left[\exp\left(\int_{(0,t]\times D(\varepsilon,\infty)} i\langle u, x\rangle) J(d(s,x))\right)\right]\right| \\ &= \left|\exp\left(\int_{(0,t]\times D(\varepsilon,\infty)} (e^{i\langle u, x\rangle} - 1)\tilde{\nu}(d(x,s))\right)\right| \\ &= \exp\left(\int_{(0,t]\times D(\varepsilon,\infty)} (\cos(\langle u, x\rangle) - 1)\tilde{\nu}(d(s,x))\right) \end{aligned}$$

Letting $\varepsilon \downarrow 0$ leads to

$$\int_{(0,t]\times D(0,\infty)} |\langle u,x\rangle|^2 \tilde{\nu}(d(s,x)) \leq$$
$$C \int_{(0,t]\times D(0,\infty)} (1-\cos(\langle u,x\rangle)\tilde{\nu}(d(s,x)) < \infty$$

for some $C > 0$. Equation (78) follows immediately since u is arbitrary.

For each $\varepsilon \in (0,1]$ consider the martingales

$$M_t^\varepsilon(\omega) = \int_{(0,t]\times D(\varepsilon,\infty)} x(J(d(s,x),\omega) - \tilde{\nu}(d(s,x)),$$

and let V_t denote the last term of (77). Repeating the proofs of Lemmas (3.9) and (3.10) with J and M^ε instead of N and S_ε respectively we have that $M_t = M_t^0$ is an a.s well defined cadlag martingale and that M_t^ε converges to M_t uniformly on bounded time intervals. Thus $M+V$ has the same jumps a.s. as X, and so $Y = X - M - V$ is a.s. a continuous additive process. By Theorem 1.33 we have that Y is a Gaussian process with continuous mean and covariance functions. Subtracting the means m_t yields a continuous, centered at 0 Gaussian process G. Lemma 3.14 implies that G and $M^\varepsilon + V$ are independent, and we extend this to M by passing to the limit as $\varepsilon \downarrow 0$. $\square$

4. Transition Functions and Markov Property

An important property of additive processes in law is the Markov property. Here we introduce Markov processes by using transition functions, and then characterize Lévy processes and additive processes as Markov processes with spatially homogeneous transition functions.

Definition 4.1. A family of mappings $P_{s,t}(x,B)$ of $x \in \mathbb{R}^d$ and $B \in \mathcal{B}(\mathbb{R}^d)$ with $0 \leq s \leq t < \infty$ is called a *system of transition functions* on $\mathbb{R}^d$ if

1) for any fixed x, it is a probability measure as a mapping of B
2) for any fixed B, it is a measurable function as a mapping of x
- $P_{s,s}(x,B) = \delta_x(B)$ for $s \geq 0$

3) it satisfies

$$\int_{\mathbb{R}^d} P_{t,u}(y,B)P_{s,t}(x,dy) = P_{s,u}(x,B) \tag{79}$$

If in addition,

5) $P_{s+h,t+h}(x,B)$ does not depend on h,

then it is called a *temporally homogeneous transition system* and we use $P_t(x,B)$ to indicate

$$P_t(x,B) = P_{s,s+t}(x,B) \qquad \text{for } s \geq 0$$

Property (4) is called the *Chapman–Kolmogorov identity.* In the case of temporally homogeneous transition system it is written as

$$\int_{\mathbb{R}^d} P_t(y,B)P_s(x,dy) = P_{s+t}(x,B) \qquad \text{for } s,t \geq 0$$

Properties (1) and (2) imply that the mapping $x \mapsto \int f(y)P_{s,t}(x,dy)$ is measurable for any bounded measurable function f, for any such f can be expressed as the limit of finite linear combinations of indicator functions of Borel sets.

Lemma 4.1. *For a transition system $P_{s,t}$ and a bounded measurable function f, we have*

$$\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} f(y)P_{t,u}(x,dy)\Big)P_{s,t}(x_0,dx) = \int_{\mathbb{R}^d} f(y)P_{s,u}(x_0,dy)$$

where $0 \leq s \leq t \leq u < \infty$.

Proof. Statement is true for $f(y) = \mathbf{1}_B(y)$, then it holds for simple functions. We now extend to any positive bounded measurable function by monotone convergence and by linearity and homogeneity to any bounded measurable function. □

If a transition systems $P_{s,t}$ on $\mathbb{R}^d$ with $0 \leq s \leq t < \infty$ are given, then for any $a \in \mathbb{R}^d$ we can construct a stochastic processes Y on a certain probability space $(\Omega^0, \mathcal{F}^0, \mathbb{P}^{0,a})$, so that $\mathbb{P}[Y_0 = a] = 1$, and such that $\mathbb{P}^{0,a}[Y_t \in B] = P_{0,t}(a,B)$. Let $\Omega^0 = (\mathbb{R}^d)^{[0,\infty)}$ with the product σ–algebra. For any $0 \leq t_0 < ... < t_n$ and $B_0, \dots, B_n$ we set

$$\mathbb{P}^{0,a}_{t_0,\dots,t_n}(B_0 \times \cdots \times B_n) = \int_{B_0}\int_{B_1}\dots\int_{B_n} P_{t_{n-1},t_n}(x_{n-1},dx_n)\dots P_{t_0,t_1}(x_0,dx_1)P_{0,t_0}(a,dx_0) \quad (80)$$

Lemma (4.1) implies that the family of multidimensional measures $\{\mathbb{P}^{0,a}_{t_0,\dots,t_n}\}$ satisfies Kolmogorov's consistency condition, thus, there is a unique probability measure $\mathbb{P}^{0,a}$ on $\mathcal{F}^0$ that extends this family. We consider now the process Y defined by the projections $Y_t(\omega) = \omega(t)$. Notice that for any $A \in \mathcal{F}^0$, the map $a \mapsto \mathbb{P}^{0,a}[A]$ is measurable. Similarly, given $s \geq 0$ and $a \in \mathbb{R}^d$, we consider the restriction $P_{t,u}(x,B)$ with $s \leq t \leq u$, $\Omega^s = (\mathbb{R}^d)^{[s,\infty)}$ and $Y_t(\omega) = \omega(t)$. We define $\mathbb{P}^{s,a}_{t_0,\dots,t_n}$ for $s \leq t_0 \dots \leq t_n$ and then obtain $\mathbb{P}^{s,a}$ on $\mathcal{F}^s$ generated by Y_t, $t \geq s$.

Definition 4.2. A stochastic process $\{X_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *Markov process with transition system $P_{s,t}$ and starting from point a at time* 0 , if it is identical in law to the process $\{Y_t : t \geq 0\}$ on $(\Omega^0, \mathcal{F}^0, \mathbb{P}^{0,a})$ defined above. If in addition, the transition system is temporally homogeneous, then X is said to be *temporally homogeneous* Markov process. In the same way, a stochastic process $\{X_t : t \geq s\}$ defined in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Markov process with transition system $P_{t,u}$ starting from a at time s, if it is identical in law to $\{Y_t : t \geq s\}$ on $(\Omega^s, \mathcal{F}^s, \mathbb{P}^{s,a})$ as defined above.

Definition 4.3. A transition system $P_{s,t}$ on $\mathbb{R}^d$ is said to be *spatially homogeneous (or translation invariant)* if

$$P_{s,t}(x, B) = P_{s,t}(0, B - x)$$

for any $0 \leq s \leq t$ $x \in \mathbb{R}^d$, where $B - x = \{y - x : y \in B\}$.

Theorem 4.2. *(i) Let X be an additive process in law on $\mathbb{R}^d$. Define $P_{s,t}$ by*

$$P_{s,t}(x, B) = \mathbb{P}[X_t - X_s \in B - x] \qquad \text{for } 0 \leq s \leq t,$$

then $P_{s,t}$ is a spatially homogeneous transition system, and X is a Markov process with transition system $P_{s,t}$ and starting at point 0 at time 0.

(ii) Conversely, if X is a stochastically continuous Markov process on $\mathbb{R}^d$ with spatially homogeneous transition system $P_{s,t}$ and starting at point 0, then X is an additive process in law.

Proof. (i) Properties (1) and (3) for transition systems are clear. Let $\mu_{s,t}$ be as in Theorem (2.18) and let $B \in \mathcal{B}(\mathbb{R}^d)$ be fixed. Then, it is clear that the mapping

$$x \mapsto P_{s,t}(x, B) = \mu_{s,t}(B - x) = \int \mathbf{1}_B(x + y)\mu_{s,t}(dy)$$

is measurable. It also follows that for any bounded measurable function f

$$\int f(y)P_{s,t}(x, dy) = \int f(x + y)\mu_{s,t}(dy) \tag{81}$$

Therefore for any $0 \leq s \leq t \leq u$

$$\begin{aligned}\int_{\mathbb{R}^d} P_{t,u}(y,B)P_{s,t}(x,dy) &= \int_{\mathbb{R}^d} P_{t,u}(x+y,B)\mu_{s,t}(dy) \\ &= \int_{\mathbb{R}^d} \mu_{t,u}(B-x-y)\mu_{s,t}(dy) \\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{B-x}(z+y)\mu_{s,t}(dy)\mu_{t,u}(dz) \\ &= (\mu_{s,t}*\mu_{t,u})(B-x) = \mu_{s,u}(B-x) = P_{s,u}(x,B)\end{aligned}$$

which is Chapman–Kolmogorov's condition. Hence, $P_{s,t}$ is a spatially homogeneous transition system. Using Fubini's theorem (n+1)–times combined with equation (81) we have that

$$\begin{aligned}&\mathbb{P}[X_{t_0} \in B_o, \ldots, X_{t_n} \in B_n] \\ = &\int_{(\mathbb{R}^d)^{n+1}} \mathbf{1}_{B_0}(x_0)\mathbf{1}_{B_1}(x_0+x_1)\cdots\mathbf{1}_{B_n}(x_0+\cdots+x_n)\cdot \\ &\qquad\cdot\mu_{t_{n-1},t_n}(dx_n)\cdots\mu_{t_0,t_1}(dx_1)\mu_{0,t_0}(dx_0) \\ = &\int_{B_0}\int_{B_1}\ldots\int_{B_n} P_{t_{n-1},t_n}(x_{n-1},dx_n)\ldots P_{t_0,t_1}(x_0,dx_1)P_{0,t_0}(0,dx_0)\end{aligned}$$

Thus, X has the same law as the process Y on $(\Omega^0, \mathcal{F}^0, \mathbb{P}^{0,a})$ constructed above with $a = 0$.
(ii) Suppose that $P_{s,t}$ is a spatially homogeneous transition system and that X is a Markov process associated to it and starting at point 0 at time 0. Let us define the system of measures $\{\mu_{s,t} : 0 \leq s \leq t < \infty\}$ by

$$\mu_{s,t}(B) = P_{s,t}(0,B)$$

Then

$$\begin{aligned}P_{s,t}(x,B) &= P_{s,t}(0,B-x) = \mu_{s,t}(B-x) \\ &= \int \mathbf{1}_B(x+y)\mu_{s,t}(dy),\end{aligned}$$

thus, equation (81) holds for any bounded measurable function. Therefore, for $0 \leq t_0 < \ldots < t_n$ we get

$$\begin{aligned}
&\mathbb{P}^{0,0}[X_{t_0} \in B_0, \ldots, X_{t_n} \in B_n] \\
= &\int_{B_0}\int_{B_1}\cdots\int_{B_n} P_{t_{n-1},t_n}(x_{n-1}, dx_n)\ldots P_{t_0,t_1}(x_0, dx_1)P_{0,t_0}(0, dx_0) \\
= &\int_{(\mathbb{R}^d)^{n+1}} \mathbf{1}_{B_0}(x_0)\mathbf{1}_{B_1}(x_0+x_1)\cdots\mathbf{1}_{B_n}(x_0+\cdots+x_n)\cdot \\
&\qquad \cdot\mu_{t_{n-1},t_n}(dx_n)\cdots\mu_{t_0,t_1}(dx_1)\mu_{0,t_0}(dx_0)
\end{aligned}$$

Hence, for any bounded measurable function f in $(\mathbb{R}^d)^n$ we have that

$$\begin{aligned}
\mathbb{E}^{0,0}[f(X_{t_0}, \ldots, X_{t_n})] = &\int_{(\mathbb{R}^d)^{n+1}} f(x_0, \ldots, x_0+\cdots+x_n)\cdot \\
&\cdot\mu_{t_{n-1},t_n}(dx_n)\cdots\mu_{t_0,t_1}(dx_1)\mu_{0,t_0}(dx_0)
\end{aligned}$$

Taking $f(x_0, \ldots, x_n) = \exp\left(i\sum_{j=1}^{n}\langle z_j, x_j - x_{j-1}\rangle\right)$ we prove that X has independent increments and that $X_t - X_s$ has distribution $\mu_{s,t}$. Hence X is an additive process in law. □

If the transition system is temporally homogeneous in addition, then we have a characterization of Lévy processes as temporally homogeneous Markov processes with spatially homogeneous transition system.

Theorem 4.3.

(1) *Let μ be an infinitely divisible distribution on $\mathbb{R}^d$, and X be a Lévy process in law corresponding to μ. Define P_t by*

$$P_t(x, B) = \mu^t(B - x).$$

Then P_t is a temporally and spatially homogeneous transition system, and X is a Markov process with transition system P_t and starting at point 0 at time 0.

(2) *Conversely, if X is any stochastically continuous, temporally homogeneous Markov process on $\mathbb{R}^d$ with spatially homogeneous transition system P_t and starting at point 0 is a Lévy process in law corresponding to the infinitely divisible distribution $\mu(\cdot) = P_1(0, \cdot)$.*

Proof. Same as in Theorem (4.2) but with $P_{s,t} = P_{t-s} = \mu^{t-s}$.

Theorem 4.4. *(Markov Property) Consider the path representation of a Markov process Y with transition system $P_{s,t}$. Let $f(y_0,\dots,y_1)$ be a bounded measurable function. Then, for any $0 \le t_0 < \dots < t_n$ it follows that the mapping $a \mapsto \mathbb{E}[f(Y_{t_0},\dots,Y_{t_n})]$ is measurable, and also that*

$$\begin{aligned}
&\mathbb{E}^{0,a}[f(Y_{t_0},\dots,Y_{t_n})] \\
= &\int\limits_{\mathbb{R}^d}\Big(\int\limits_{\mathbb{R}^d}\cdots\Big(\int\limits_{\mathbb{R}^d} f(y_0,\dots,y_0+\dots+y_n)P_{t_{n-1},t_n}(y_{n-1},dy_n)\Big) \\
&\qquad\qquad \cdots P_{t_0,t_1}(y_0,dy_1)\Big)P_{0,t_0}(a,dy_0) \qquad (82)
\end{aligned}$$

Moreover, for any $0 \le s_0 < \dots < s_m \le s$ and for any bounded measurable function $g(y_o,\dots,y_m)$ we have

$$\mathbb{E}^{0,a}[g(Y_{s_0},\dots,Y_{s_m})f(Y_{s+t_0},\dots,Y_{s+t_n})] \qquad (83)$$

$$= \mathbb{E}^{0,a}\Big[g(Y_{s_0},\dots,Y_{s_m})\mathbb{E}^{s,Y_s}[f(Y_{s+t_0},\dots,Y_{s+t_n})]\Big] \qquad (84)$$

Property (83) is referred as the *Markov property.*

Proof. Equation (82) has already been proved for functions of the form $\mathbf{1}_{B_0}(y_0)\cdots\mathbf{1}_{B_n}(y_n)$. The extension to a more general bounded measurable function g is straightforward.
For equation (83), note first that $\mathbb{E}^{s,Y_s}[f(Y_{s+t_0},\dots,Y_{s+t_n})] = h(Y_s)$ where

$$h(x) = \mathbb{E}^{s,x}[f(Y_{s+t_0},\dots,Y_{s+t_n})]$$

Let

$$\begin{aligned}
g(y_0,\dots,y_n) &= \mathbf{1}_{C_0}(y_0)\cdots\mathbf{1}_{C_m}(y_n) \\
f(y_0,\dots,y_n) &= \mathbf{1}_{B_0}(y_0)\cdots\mathbf{1}_{B_n}(y_n)
\end{aligned}$$

Then the left hand side of (83) equals

$$\mathbb{P}[Y_{s_0}\in C_0,\dots,Y_{s_m}\in C_m, Y_s\in\mathbb{R}^d,\dots,Y_{s+t_0}\in B_0,\dots,Y_{s+t_n}\in B_n]$$

We then rewrite the former probability by the transition function as in (80), and integrate $(n+1)$ times and use (82) twice. Then we get (83) for this special forms of g and f. Extension to measurable bounded functions g and f is then straightforward. □

Lévy processes have a stronger property than the Markov property.

Proposition 4.5. *Let X be a Lévy process on $\mathbb{R}^d$. Then for any $s \ge 0$, the process Y defined by $Y_t = X_{s+t} - X_s$ is a Lévy process identical in law with X. Also $\{Y_t : t \ge 0\}$ and $\{X_{s+t} - X_s : 0 \le t \le s\}$ are independent.*

Proof. Since for fixed $s \geq 0$, $Y_0 = 0$ and $Y_{t_2} - Y_{t_1} = X_{t_2} - X_{t_1}$, it follows from the definition of Lévy process that Y is also a Lévy process and X and Y are equal in law. The rest is immediate. □

Remark 4.1. Sometimes it is useful to consider a random starting point. Given a transition system $P_{s,t}$, let Y and $(\Omega^0, \mathcal{F}^0, \mathbb{P}^{0,a})$ as before. For any probability measure ρ on $\mathbb{R}^d$, define

$$\mathbb{P}^{0,\rho}[A] = \int_{\mathbb{R}^d} \mathbb{P}^{0,a}[A] \rho(da).$$

A stochastic process X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Markov process with transition system $P_{s,t}$ and *initial distribution* ρ, if it is identical in law with the process Y on the probability space $(\Omega^0, \mathcal{F}^o, \mathbb{P}^{0,\rho})$.

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