BASIC TOPICS IN HARMONIC ANALYSIS I: FOURIER TRANSFORM AND PLANCHEREL THEOREM

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ABSTRACT. These lecture notes discuss the $\mathcal{L}_1(\mathbb{R}^n)$ and $\mathcal{L}_2(\mathbb{R}^n)$ theory of the Fourier transform. We make a brief treatment of tempered distributions and use it to classify the operators in $\mathcal{L}_1(\mathbb{R}^n)$ and in $\mathcal{L}_2(\mathbb{R}^n)$ that commute with translations.

1. Introduction

The objective of theses notes is to have a self-contained and short reference to the fundamental results in harmonic analysis and the techniques used to prove them. All this results presented here are classical and appear in many textbooks [Rud87, Ste70, SW71]. We made the effort to present as many details in the proofs as possible. We hope that these notes are helpful to graduate students who are learning the foundations of analysis.

2. $\mathcal{L}_1(\mathbb{R}^n)$ theory of the Fourier transform

We begin by defining the Fourier transform in $\mathcal{L}_1(\mathbb{R}^n)$.

Definition 2.1. If $f \in \mathcal{L}_1(\mathbb{R}^n)$, the Fourier transform of f is the function $\hat{f}$ defined by letting

$$\hat{f}(t) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot t}\, dx$$

The following properties of the Fourier transform are easy to obtain.

Proposition 2.1. *Suppose $f \in \mathcal{L}_1(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. Then,*

(a) *If $g(x) = f(x)e^{2\pi i x\cdot h}$ then $\hat{g}(t) = \hat{f}(t-h)$.*
(b) *If $g(x) = f(x-h)$ then, $\hat{g}(t) = \hat{f}(t)e^{-2\pi i h\cdot t}$.*
(c) *If $g \in \mathcal{L}_1(\mathbb{R}^n)$ and $\varphi = f * g$ then $\hat{\varphi}(t) = \hat{f}(t)\hat{g}(t)$.*
(d) *If $g(x) = \overline{f(-x)}$, then $\hat{g}(t) = \overline{\hat{f}(t)}$.*
(e) *If $g(x) = f(x/\alpha)$ and $\alpha > 0$, then $\hat{g}(t) = \alpha^n \hat{f}(\alpha t)$.*

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Proof. We only prove *(c)*. The rest are straight forward.

$$\begin{aligned}
\hat{\varphi}(t) &= \int\limits_{\mathbb{R}^n} e^{-2\pi i x\cdot t}\Big(\int\limits_{\mathbb{R}^n} f(x-y)g(y)\,dy\Big)\,dx \\
&= \int\limits_{\mathbb{R}^n} e^{-2\pi y\cdot t}g(y)\Big(\int\limits_{\mathbb{R}^n} f(x-y)e^{-2\pi i(x-y)\cdot t}\,dx\Big)\,dy \\
&= \int\limits_{\mathbb{R}^n} e^{-2\pi i y\cdot t}g(y)\Big(\int\limits_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot t}\,dx\Big)\,dy \\
&= \hat{f}(t)\hat{g}(t)
\end{aligned}$$

□

For any positive number a and any vector h we define the dilation by a, δ_a, and the translation by h, τ_h, as the operators mapping any function $g(x)$ into $g(ax)$ and $g(x-h)$ respectively. Proposition 2.1 formulated in terms of these operators says

(a') $(e^{2\pi i x\cdot h}f(x))^\wedge(t) = (\tau_h\hat{f})(t)$.
(b') $(\tau_h g)^\wedge(t) = e^{-2\pi i t\cdot h}\hat{f}(t)$.
(e') $(\delta_a f)^\wedge(t) = a^{-n}\hat{f}(a^{-1}t)$.

The following lemma will be needed:

Lemma 2.2. *If $1 \le p < \infty$, then the mapping τ from $\mathbb{R}$ to $\mathcal{L}_p(\mathbb{R}^n)$ given by $h \mapsto \tau_h f$ is uniformly continuous.*

Proof. We first prove this lemma for continuous functions of compact support. Suppose that g is such a function and that $\text{supp}(g) \subset B(0,a)$ then, g is uniformly continuous. Given $\varepsilon > 0$, by uniform continuity of there is a $0 < \delta < a$ such that $|s-t| < \delta$ implies

$$|g(t) - g(t)| < (m(B(0,3a)))^{-1/p}\varepsilon$$

Thus

$$\int\limits_{\mathbb{R}^n} |g(x-t) - g(x-s)|^p\,dx < \varepsilon^p$$

so that $\|\tau_t g - \tau_s g\|_p = \|\tau_{t-s}g - g|_p < \varepsilon$. The conclusion follows from the density of $\mathcal{C}_c(\mathbb{R}^n)$ in $\mathcal{L}_p(\mathbb{R}^n)$. □

Another useful result of this nature, which we will use until the next section, has to do with convolution of functions.

Theorem 2.3. *If $1/p + 1/q = 1$, $f \in \mathcal{L}_p$ and $g \in \mathcal{L}_q$, then $f * g$ is uniformly continuous. If $1 < p < \infty$ then $\lim\limits_{|x|\to\infty} f * g(x) = 0$.*

Proof. Without lost of generality, we might assume that $1 \leq p < \infty$, then by Hölder's inequality and translation invariance of Lebesgue measure we have

$$\begin{aligned} |(f*g)(x+h)-(f*g)(x)| &\leq \int |(f(x+h-y)-f(x-y)||g(y)|\,dy \\ &\leq \|\tau_{-h}f-f\|_p\|g\|_q \end{aligned}$$

Lemma 2.2 implies uniform continuity. To prove that $f*g$ vanishes at infinity, we use sequences f_k and g_k of compact supported functions approximating f and g. If $\operatorname{supp} f_k \cup \operatorname{supp} g_k \subset B_{a_k}(0)$, then $f_k * g_k$ is continuous and of compact support, $\operatorname{supp}(f_k * g_k) \subset B_{2a_k}(0)$. Using Hölder's inequality we get that

$$\|f*g-f_k*g_k\|_\infty \leq \|f-f_k\|_p\|g\|_q + \|f_k\|_p\|g-g_k\|_q$$

The conclusion of the Theorem follows immediately. □

It is easy now to establish and prove the following result:

Theorem 2.4. (Riemann -Lebesgue) *(a) The mapping $f \mapsto \hat{f}$ is a bounded linear transformation from $\mathcal{L}_1(\mathbb{R}^n)$ to $\mathcal{L}_\infty(\mathbb{R}^n)$. In fact $\|\hat{f}\|_\infty \leq \|f\|_1$. (b) If $f \in \mathcal{L}_1(\mathbb{R}^n)$ then $\hat{f}$ is uniformly continuous and $\hat{f}(t) \to 0$ as $|t| \to \infty$.*

Proof. We only prove (b). Note that

$$|\hat{f}(t)-\hat{f}(s)| \leq \int_{\mathbb{R}^n} |f(x)||e^{-2\pi i x\cdot(t-s)}-1|\,dx$$

Dominated convergence does the rest. To prove that $\hat{f}$ vanishes at infinity, note that since $e^{\pi i} = -1$ then

$$\hat{f}(t) = -\int f(x)e^{-2\pi i\left(x+\frac{t}{2|t|^2}\right)\cdot t}\,dx = -\int f\left(x-\tfrac{t}{2|t|^2}\right)e^{-2\pi i x\cdot t}\,dx$$

Hence

$$2\hat{f}(t) = \int \left(f(x)-f\left(x-\tfrac{t}{2|t|^2}\right)\right)e^{-2\pi i x\cdot t}\,dx$$

so that

$$2\hat{f}(t) \leq \|f-\tau_h f\|_1$$

with $h = \frac{t}{2|t|^2}$. Now, lemma 2.2 implies that $\hat{f}(t) \to 0$ as $|t| \to \infty$. □

Differentiation and Fourier transformation are related in the following way:

Theorem 2.5. *Suppose $f \in \mathcal{L}_1(\mathbb{R}^n)$ and $x_k f(x) \in \mathcal{L}_1(\mathbb{R}^n)$, where x_k is the r-th coordinate function. Then $\hat{f}$ is differentiable with respect to t_k and*

$$\frac{\partial \hat{f}}{\partial t_k}(t) = (-2\pi i x_k f(x))^\wedge(t).$$

Proof. Letting $h = (0, \dots, h_k, \dots, 0)$ be a nonzero vector along the k-th axis. Since $|e^{-2\pi i t_k h_k} - 1| \leq t_k h_k$, we have that

$$\frac{\hat{f}(t+h) - \hat{f}(t)}{h_k} = \int e^{-2\pi i x\cdot t} \frac{e^{-2\pi i x\cdot h} - 1}{h_k} f(x)\, dx \to (-2\pi i x_k f(x))\hat{}\,(t)$$

as $h_k \to 0$ by dominated convergence. □

It is also true that we can take Fourier transforms of partial derivatives of functions. To make precise this statement, we introduce the following

Definition 2.2. We say that $f \in \mathcal{L}_p(\mathbb{R}^n)$ is *differentiable in the* $\mathcal{L}_p(\mathbb{R}^n)$ *norm with respect to* x_k, if there exists $g \in \mathcal{L}_p(\mathbb{R}^n)$ such that

$$\Big\| \frac{f(x+h) - f(x)}{h_k} - g(x) \Big\|_p \to 0$$

as $h_k \to 0$

Applying proposition 2.1 and part *(a)* of theorem 2.4 leads to

$$\Big| \frac{e^{2\pi i t\cdot h} - 1}{h_k} \hat{f}(t) - \hat{g}(t) \Big| \leq \Big\| \frac{f(x+h) - f(x)}{h_k} - g(x) \Big\|_1$$

and taking $h_k \to 0$ proves the following

Theorem 2.6. *If* $f \in \mathcal{L}_1(\mathbb{R}^n)$ *and* g *is the partial derivative of* f *with respect to* x_k *in the* $\mathcal{L}_1(\mathbb{R}^n)$ *norm then*

$$\hat{g}(t) = 2\pi i t_k \hat{f}(t)$$

everywhere.

For $n = 1$ we can use integration by parts to get a simpler result.

Proposition 2.7. *If* f *and* f' *belong to* $L^1(\mathbb{R})$ *and* f *is the indefinite integral of* f', *then* $(f')\hat{}\,(t) = 2\pi i t \hat{f}(t)$

Proof. Since $f(x) = \int\limits_{-\infty}^{x} f'(t)\, dt$, it follows that

$$\lim_{x\to-\infty} f(x) = 0 = \lim_{x\to\infty} f(x).$$

The first assertion is straight forward. To check the second equality observe that $\lim_{x\to\infty} f(x) = \int f'(t)\, dt$ by dominated convergence. If $b = \int f'(t)\, dt \neq 0$, then there exists $A > 0$ such that $|f(x)| > |b|/2$ for all $x \geq A$. Hence $\int_{\{|f|>|b|/2\}} |f(x)|\, dx \geq \int_{[A,\infty)} |f(x)|\, dx = \infty$, which contradicts the assumption $f \in \mathcal{L}_1$. Therefore $b = 0$.

The conclusion the proposition follows immediately from integration by parts. □

The last two theorems can be extended to higher derivatives. Without going into the details, we note the following formulas:

$$\begin{array}{lll} \text{(i)} & P(D)\hat{f}(t) & = (P(-2\pi i x) f(x))\hat{}\,(t) \\ \text{(ii)} & (P(D)f)\hat{}\,(t) & = P(2\pi i t)\hat{f}(t) \end{array} \tag{2.1}$$

where, for a n n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ of nonnegative integers we let $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, $D^\alpha = \partial^{\alpha_1+\alpha_2+\ldots+\alpha_n} / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}$, P is a polynomial in the n variables $x_1, x_2, \ldots, x_n$ and $P(D)$ is the associated differential operator.
The main problem in the $\mathcal{L}_1(\mathbb{R}^n)$ theory of Fourier transform is to obtain a function f back from its Fourier transform $\hat{f}$. Our next task is to answer this question. Here we introduce certain *summability* methods for integrals.

Definition 2.3. For each $\varepsilon > 0$ and f a locally integrable function, we define the Abel mean $A_\varepsilon f$ to be the integral

$$A_\varepsilon f = \int\limits_{\mathbb{R}^n} f(x) e^{-\varepsilon |x|} \, dx \tag{2.2}$$

In the same way, we define the Gauss-Weierstrass mean $W_\varepsilon f$ as

$$W_\varepsilon f = \int\limits_{\mathbb{R}^n} f(x) e^{-\varepsilon |x|^2} \, dx. \tag{2.3}$$

Whenever the limit $\lim\limits_{\varepsilon \to 0} A_\varepsilon f$ or $\lim\limits_{\varepsilon \to 0} W_\varepsilon f$ exists, we say that $\int_{\mathbb{R}^n} f(x) \, dx$ is Abel or Weierstrass summable.

From dominated convergence shows that if f is integrable, then $\lim\limits_{\varepsilon \to 0} A_\varepsilon f = \int_{\mathbb{R}^n} f(x) \, dx$. The following examples shows that a function f might be Abel or Weierstrass summable without being integrable.
Example 1 Let f be the function

$$f(x) = \frac{\sin x}{x} \mathbf{1}_{(0,\infty)}(x)$$

We claim that $A f = \lim\limits_{\varepsilon \to 0} A_\varepsilon f = \pi/2$ For any $\delta > 0$ there exist $P > 0$ such that

$$\Big| \int\limits_0^p \frac{\sin x}{x} \, dx - \frac{\pi}{2} \Big| < \frac{\delta}{3}; \qquad \int\limits_p^\infty e^{-x} \, dx < \frac{\delta}{3}$$

Since f is integrable on the interval $[0, P]$ then there exists $\gamma > 0$ such that whenever $0 < \varepsilon < \gamma$ then

$$\Big| \int\limits_0^p (e^{-\varepsilon x} - 1) \frac{\sin x}{x} \, dx \Big| < \frac{\delta}{3}$$

Combining these three inequaliti we prove our statement.
Equations (2.2) and (2.3) can be put in a more general framework

$$M_{\varepsilon,\varphi} f = M_\varepsilon f = \int\limits_{\mathbb{R}^n} \varphi(\varepsilon x) f(x) \, dx$$

where $\varphi \in \mathcal{C}_0$ and $\varphi(0) = 1$. Then $\int_{\mathbb{R}^n} f(x)$ is summable to l if $\lim\limits_{\varepsilon \to 0} M_\varepsilon f = l$.
We shall call $M_{\varepsilon,\varphi} f$ the φ means of this integral.

The following computational results will be needed in our pursue of "inverting" the Fourier transform.

Lemma 2.8. *If* $\varphi(x) = e^{-\pi x^2}$*, then* $\hat{\varphi}(t) = \varphi(t)$

Proof. We give a simple ODE proof of this fact. First note that

$$\hat{\varphi}(0) = \int_{\mathbb{R}} e^{-\pi x^2}\, dx = 1$$

It is easy to see that φ satisfies the following initial value problem

$$\varphi\prime(x) + 2\pi x\varphi(x) = 0 \quad \varphi(0) = 1$$

If we apply theorem 2.5 and proposition 2.7, then by taking Fourier transform on the last equation we get that $\hat{\varphi}$ satisfies same initial value problem as φ. Therefore $\hat{\varphi}(t) = e^{-\pi t^2}$. □

Next, we will need to establish the following identity:

$$e^{-\beta} = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\beta^2/4u}\, du \tag{2.4}$$

To do so, we will use the following identities

$$e^{-\beta} = \frac{2}{\pi}\int_0^\infty \frac{\cos\beta x}{1+x^2}\, dx, \quad \text{with } \beta > 0$$

and

$$\frac{1}{1+x^2} = \int_0^\infty e^{-(1+x^2)u}\, du$$

The second of this identity is obvious, while the first is an easy application of the theory of residues to the function $e^{i\beta z}/(1+z^2)$. Then

$$\begin{aligned}
e^{-\beta} &= \frac{2}{\pi}\int_0^\infty \frac{\cos\beta x}{1+x^2}\, dx = \frac{2}{\pi}\int_0^\infty \cos\beta x \Big(\int_0^\infty e^{-u}e^{-ux^2}\, du\Big)\, dx \\
&= \frac{2}{\pi}\int_0^\infty e^{-u}\Big(\int_0^\infty e^{-ux^2}\cos\beta x\, dx\Big)\, du \\
&= \frac{1}{\pi}\int_0^\infty e^{-u}\Big(\int_{-\infty}^\infty e^{-ux^2}e^{-i\beta x}\, dx\Big)\, du \\
&= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\beta^2/4u}\, du
\end{aligned}$$

Now we are in conditions to state and prove the following

Theorem 2.9. *For all* $\alpha > 0$ *we have that*

$$\int_{\mathbb{R}^n} e^{-2\pi i y\cdot t} e^{-\pi\alpha|y|^2}\,dy = \alpha^{-n/2} e^{-\pi|t|^2/\alpha} \tag{2.5}$$

and

$$\int_{\mathbb{R}^n} e^{-2\pi i y\cdot t} e^{-2\pi\alpha|y|}\,dy = c_n \frac{\alpha}{(\alpha^2+|t|^2)^{(n+1)/2}} \tag{2.6}$$

where $c_n = \Gamma[(n+1)/2]/(\pi^{(n+1)/2})$.

Proof. By a change of variables, it suffices to consider the case $\alpha = 1$. Equality (2.5) follows easily from lemma 2.8. Since

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-2\pi i y\cdot t} e^{-\pi|y|^2}\,dy &= \prod_{j=1}^{n} \int_{-\infty}^{\infty} e^{-2\pi i y_j t_j} e^{-\pi y_j^2}\,dy_j \\ &= \prod_{j=1}^{n} e^{-\pi t_k^2} = e^{-\pi|t|^2}\end{aligned}$$

Equation (2.6) is harder to obtain. Using equation (2.4) we get

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-2\pi i y\cdot t} e^{-2\pi|y|}\,dy &= \int_{\mathbb{R}^n} \Big(\frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-u}}{\sqrt{u}} e^{-\pi^2|y|^2/u}\,du\Big) e^{-2\pi i y\cdot t}\,dy \\ &= \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-u}}{\sqrt{u}} \Big(\int_{\mathbb{R}^n} e^{-\pi^2|y|^2/u} e^{-2\pi i y\cdot t}\,dy\Big)\,du \\ &= \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-u}}{\sqrt{u}} \Big(\big(\sqrt{\frac{u}{\pi}}\big)^{n/2} e^{-u|t|^2}\Big)\,du \\ &= \frac{1}{\pi^{(n+1)/2}} \int_0^{\infty} e^{-u} u^{(n-1)/2} e^{-u|t|^2}\,du \\ &= \frac{1}{\pi^{(n+1)/2}} \frac{1}{(1+|t|^2)^{(n+1)/2}} \int_0^{\infty} e^{-s} s^{(n-1)/2}\,ds \\ &= \frac{\Gamma[(n+1)/2]}{\pi^{(n+1)/2}} \frac{1}{(1+|t|^2)^{(n+1)/2}}\end{aligned}$$

For $\alpha > 0$, let us denote the Fourier transform of the functions $\varphi_\alpha(x) = e^{-4\pi\alpha|x|^2}$ and $\rho_\alpha(x) = e^{-2\pi\alpha|x|}$ by $W(t,\alpha)$ and $P(t,\alpha)$ respectively. That is,

$$\begin{aligned} W(t,\alpha) &= \frac{1}{(4\pi\alpha)^{n/2}} e^{-|t|^2/4\alpha} \\ P(t,\alpha) &= c_n \frac{\alpha}{(\alpha^2 + |t|^2)^{(n+1)/2}} \end{aligned}$$

$W(t,\alpha)$ is called the Gauss–Weierstrass kernel. $P(t,\alpha)$ is called the Poisson kernel. We will show among other things that the Abel and Gaussian means of the integral

$$\int_{\mathbb{R}^n} \hat{f}(t) e^{2\pi i t\cdot x}\, dt \tag{2.7}$$

converges almost everywhere to $f(x)$. The idea will be to expressed those means in terms of convolutions with the Poisson and Gauss kernels and then use the theory of approximations to the identity. for the first step towards this, we will use the following result.

Theorem 2.10. *If f anf g belong to $\mathcal{L}_1(\mathbb{R}^n)$ then*

$$\int_{\mathbb{R}^n} \hat{f}(t) g(t)\, dt = \int_{\mathbb{R}^n} f(t)\hat{g}(t)\, dt$$

Proof. Applying Fubini's theorem, we get

$$\begin{aligned} \int_{\mathbb{R}^n} \hat{f}(t) g(t)\, dt &= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot t}\, dx \Big) g(t)\, dt \\ &= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} g(t) e^{-2\pi i x\cdot t}\, dt \Big) f(x)\, dx \\ &= \int_{\mathbb{R}^n} f(x)\hat{g}(x)\, dx \end{aligned}$$

□

For any given function f we let $f_\varepsilon(x) = \varepsilon^{-n} f(\varepsilon^{-1}x)$. In this notation, proposition 2.1 says that $(\delta_\varepsilon\varphi)^\wedge(t) = \hat{\varphi}_\varepsilon(t)$.

Example 2 Consider the function

$$\varphi(x) = e^{-4\pi^2|x|^2}$$

its Fourier transform is given by

$$\hat{\varphi}(t) = (4\pi)^{-n/2} e^{-|t|^2/4}$$

Thus

$$(\delta_\varepsilon\varphi)(x))^\wedge(t) = (4\pi\varepsilon^2)^{-n/2} e^{-|t|^2/4\varepsilon^2} = W(t,\varepsilon^2)$$

Example 3 Consider the function

$$\rho(x) = e^{-2\pi|x|}$$

its Fourier transsform is given by

$$\hat{\varphi}(t) = c_n \frac{1}{(1+|t|^2)^{(n+1)/2}}$$

Thus

$$(\delta_\varepsilon \rho)(x))^\wedge(t) = c_n \frac{\varepsilon}{(\varepsilon^2+|t|^2)^{(n+1)/2}} = P(t,\varepsilon)$$

These examples together with theorem 2.10 lead to the following

Theorem 2.11. *If f and φ belong to $\mathcal{L}_1(\mathbb{R}^n)$ then*

$$\int\limits_{\mathbb{R}^n} \hat{f}(x) e^{2\pi i t\cdot x} \varphi(\varepsilon x)\, dx = \int\limits_{\mathbb{R}^n} f(x) \hat{\varphi}_\varepsilon(x-t)\, dx$$

for all $\varepsilon > 0$. In particular,

$$\int\limits_{\mathbb{R}^n} \hat{f}(x) e^{2\pi i t\cdot x} e^{-2\pi\varepsilon|x|}\, dx = \int\limits_{\mathbb{R}^n} f(x) P(x-t,\varepsilon)\, dx$$

and

$$\int\limits_{\mathbb{R}^n} \hat{f}(x) e^{2\pi i t\cdot x} e^{-4\pi^2\varepsilon|x|^2}\, dx = \int\limits_{\mathbb{R}^n} f(x) W(x-t,\varepsilon)\, dx$$

We have almost everything set up to use the theory of approximations to the identity to show that integral (2.7) is summable to f for a large class of methods that includes both the Able and Gauss summability. The task will be to prove that the means $\int_{\mathbb{R}^n} \hat{f}(t) e^{2\pi i t\cdot x} \varphi(\varepsilon t)\, dt$ converge to f in $\mathcal{L}_1(\mathbb{R}^n)$ provided that both φ and $\hat{\varphi}$ are integrable and $\int_{\mathbb{R}^n} \hat{\varphi}(t)\, dt = 1$. The following lemma shows that the Gauss and Poisson kernels satisfy that condition.

Lemma 2.12. *For all $\alpha > 0$*

$$\int\limits_{\mathbb{R}^n} W(x,\alpha)\, dx = 1$$

and

$$\int\limits_{\mathbb{R}^n} P(x,\alpha)\, dx = 1$$

Proof. For $\alpha > 0$, a simple change of variable gives that

$$\int_{\mathbb{R}^n} W(x,\alpha)\, dx = \int_{\mathbb{R}^n} W(x, 1/4\pi)\, dx = \int_{\mathbb{R}^n} e^{-\pi|x|^2}\, dx = 1$$

Similarly, we have that

$$\int_{\mathbb{R}^n} P(x,\alpha)\, dx = \int_{\mathbb{R}^n} P(x,1)\, dx$$

. The following geometrical facts will be at handy in our calculations:

(i) The *surface element* $d\sigma_{n-1}$ of the sphere S^{n-1} on the space $\mathbb{R}^n$ is given by:

$$\begin{aligned} d\sigma_{n-1} &= \sin^{n-2}\theta_1\, d\theta_1\, d\sigma_{n-2} \\ &= \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2}\, d\theta_1\, d\theta_2 \cdots d\theta_{n-1} \end{aligned}$$

where $0 \le \theta_k < \pi$ for $1 \le k \le n-2$ and $0 \le \theta_{n-1} < 2\pi$

(ii) The volume ν_n of the unit ball in $\mathbb{R}^n$ and the surface σ_{n-1} of its boundary are:

$$\begin{aligned} \nu_n &= \int\limits_{|x|\le 1} dx = \frac{\sigma_{n-1}}{n} \\ \sigma_{n-1} &= \int\limits_{S^{n-1}} d\sigma_{n-1}(u) = \frac{2\pi^{n/2}}{\Gamma[n/2]} \end{aligned}$$

With this in mind, if we use polar coordinates and then the change of variable $r = \tan\theta$ and we will have

$$\begin{aligned} \int\limits_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^{(n+1)/2}}\, dx &= \int\limits_0^\infty \int\limits_{S^{n-1}} \frac{r^{n-1}}{(1+r^2)^{(n+1)/2}}\, d\sigma_{n-1}(u)\, dr \\ &= \sigma_{n-1} \int\limits_0^\infty \frac{r^{n-1}}{(1+r^2)^{(n+1)/2}}\, dr \\ &= \sigma_{n-1} \int\limits_0^{\pi/2} \sin^{n-1}\theta\, d\theta \\ &= \frac{1}{2}\sigma_n = \frac{\pi^{(n+1)/2}}{\Gamma[(n+1)/2]} \end{aligned}$$

This finishes the proof. □

Now we state and proof a theorem concerning approximations to the identity

Theorem 2.13. (Approximation to the Identity) *Suppose* $\varphi \in \mathcal{L}_1$ *with* $a = \int_{\mathbb{R}^n} \varphi(x)\, dx$, *and for* $\varepsilon > 0$ *let* $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$. *If* $f \in \mathcal{L}_p$, $1 \le p < \infty$, *or* $f \in \mathcal{C}_0(\mathbb{R}^n) \subset \mathcal{L}_\infty$, *then* $\|f * \varphi_\varepsilon - a\, f\|_p \to 0$ *as* $\varepsilon \to 0$. *In particular,* $u(x,\varepsilon) = \int_{\mathbb{R}^n} f(t)P(x-t,\varepsilon)\, dt$ *and* $s(x,\varepsilon) = \int_{\mathbb{R}^n} f(t)W(x-t,\varepsilon)\, dt$ *converge to* f *in* $\mathcal{L}_p$ *as* $\varepsilon \to 0$.

Proof. By a simple change of variables we have

$$\int\limits_{\mathbb{R}^n} \varphi_\varepsilon(s)\, dx = \int\limits_{\mathbb{R}^n} \varepsilon^{-n}\varphi(\varepsilon^{-1}x)\, dx = \int\limits_{\mathbb{R}^n} \varphi(x)\, dx = a$$

Hence,

$$\begin{aligned}(f*g)(x)-af(x) &= \int_{\mathbb{R}^n}[f(x-t)-f(x)]\varphi_\varepsilon(t)\,dt\\ &= \int_{\mathbb{R}^n}[f(x-\varepsilon t)-f(x)]\varphi(t)\,dt\end{aligned}$$

For $1\le p<\infty$ let $S=\{h\in\mathcal{L}_q(\mathbb{R}^n):\|h\|_q=1\}$, where $1/p+1/q=1$. Using Minkowski's duality equation, Fubini's theorem and Hölder inequality we get

$$\begin{aligned}\|f*g-af\|_p &= \sup_S\Big(\Big|\int_{\mathbb{R}^n}h(x)\Big(\int_{\mathbb{R}^n}[f(x-\varepsilon t)-f(x)]\varphi(t)dt\Big)dx\Big|\Big)\\ &\le \sup_S\Big(\int_{\mathbb{R}^n}|h(x)|\Big(\int_{\mathbb{R}^n}|f(x-\varepsilon t)-f(x)||\varphi(t)|dt\Big)dx\Big)\\ &\le \sup_S\Big(\int_{\mathbb{R}^n}|\varphi(t)|\Big(\int_{\mathbb{R}^n}|f(x-\varepsilon t)-f(x)||h(x)|dx\Big)dt\Big)\\ &\le \int_{\mathbb{R}^n}|\varphi(t)|\|h\|_q\|\tau_{\varepsilon t}f-f\|_p\,dt\\ &= \int_{\mathbb{R}^n}|\varphi(t)|\|\tau_{\varepsilon t}f-f\|_p\,dt\end{aligned}$$

Let $\omega_{f,p}(h)=\omega(h)=\|\tau_h f-f\|_p$. Then $|\omega(h)|\le 2\|f\|_p$ for all h; consequently, by Lemma 2.2, $\omega(h)\to 0$ as $h\to 0$. Therefore, by dominated convergence $\lim_{\varepsilon\to 0} f*\varphi_\varepsilon = f$ in $\mathcal{L}_p$.

For $f\in\mathcal{C}_0(\mathbb{R}^n)$ the proof is much easier. □

Corollary 2.14. *Suppose* $\mathbf{A}(\varepsilon)$ *is a family of invertible matrices such that* $\lim_{\varepsilon\to 0}\mathbf{A}(\varepsilon)=\mathbf{0}$. *Let us define* $\varphi_{\mathbf{A}(\varepsilon)}(x)=\Big(\det\mathbf{A}(\varepsilon)\Big)^{-n}\varphi(\mathbf{A}^{-1}(\varepsilon)x)$. *Then* $\|f*\varphi_{\mathbf{A}(\varepsilon)}-a\,f\|_p\to 0$.

An important application of Theorems 2.11 and 2.13 is the solution to the Fourier inverse problem

Theorem 2.15. *Assume that* φ *and its Fourier transform* $\hat\varphi$ *are integrable, and that* $\int_{\mathbb{R}^n}\hat\varphi(x)\,dx=1$. *Then, for any integrable function* f, *the integral* $\int_{\mathbb{R}^n}\hat f(t)e^{2\pi i x\cdot t}\varphi(\varepsilon t)\,dt$ *converges to* $f(x)$ *in* $\mathcal{L}_1(\mathbb{R}^n)$ *norm as* $\varepsilon\to 0$. *In particular, the Abel and Gauss means of this integral converges to* $f\in\mathcal{L}_1(\mathbb{R}^n)$.

Since $s(x,\varepsilon)=\int_{\mathbb{R}^n}\hat f(t)e^{2\pi i x\cdot t}e^{-4\pi^2\varepsilon|t|^2}\,dt$ converges to $f(x)$ in $\mathcal{L}_1(\mathbb{R}^n)$ as $\varepsilon\to 0$ there is a sequence $\varepsilon_k\to 0$ such that $s(x,\varepsilon_k)\to f(x)$ for a.e. x. If $\hat f$ happens to be integrable as well, then by dominated convergence we get the following result:

Corollary 2.16. *If both f ad $\hat{f}$ are integrable then*

$$f(x) = \int\limits_{\mathbb{R}^n} \hat{f}(t)e^{2\pi i t\cdot x}\,dt$$

for almost every x.

Remember that $\hat{f}$ is continuous, and if $\hat{f}$ is integrable then the integral $\int_{\mathbb{R}^n} \hat{f}(t)e^{2\pi i t\cdot x}\,dt$ will be continuous as well (in fact, it is $(\hat{f})^\wedge(-x)$). Hence, by changing f on a set of measure zero we can obtain identity in Corollary 2.16. In other words, f can be make into a continuous function by changing its values on a set of measure zero.

If $\hat{f}(t) = 0$ then $f(x) = 0$ a.e. Applying this to $f_1 - f_2$ we obtain the following result on uniqueness of the Fourier transform.

Corollary 2.17. *If f_1 and f_2 belong to $\mathcal{L}_1(\mathbb{R}^n)$ and $\hat{f}_1(t) = \hat{f}_2(t)$ for all $t \in \mathbb{R}^n$ then $f_1(x) = f_2(x)$ a.e.*

The Fourier inversion problem has a pointwise solution as well. The next result is a pointwise version of the theorem of approximations to the identity. for any given measurable function φ, we defined the *least decreasing radial majorant* of φ as the function $\psi(x) = \|\mathbb{1}_{B^c(0;\|x\|)}\varphi\|_\infty$.

Theorem 2.18. *Suppose $\varphi \in \mathcal{L}_1(\mathbb{R}^n)$ with $a = \int_{\mathbb{R}^n} \varphi(x)\,dx$. If $\psi \in \mathcal{L}_1(\mathbb{R}^n)$ and $f \in \mathcal{L}_p(\mathbb{R}^n)$ $1 \leq p < \infty$, then*

(i) $\lim\limits_{\varepsilon\to 0}(f * \varphi_\varepsilon)(x) = af(x)$ *whenever x is a Lebesgue point of f.*
(ii) $\sup\limits_{\varepsilon>0} |(f * \varphi_\varepsilon)(x)| \leq \|\psi\|_1\, M_f(x)$, *where M_f is the Hardy's maximal function of f*

In particular, the Poisson integral and Gauss–Weierstrass integrals

$$u(t,\varepsilon) = \int_{\mathbb{R}^n} f(x)P(t-x,\varepsilon)\,dx, \quad s(t,\varepsilon) = \int_{\mathbb{R}^n} f(x)W(t-x,\varepsilon)\,dx$$

converge to $f(x)$ as $\varepsilon \to 0$ at every Lebesgue point x of f, and

$$\sup_{\varepsilon>0} |u(t,\varepsilon)| \leq M_f(t), \quad \sup_{\varepsilon>0} |s(t,\varepsilon)| \leq M_f(t)$$

Proof. Let x be a Lebesgue point of f. This means that for any $\delta > 0$ there exists $\eta > 0$ such that $\frac{1}{m(B_r(x))}\int_{B_r(x)} |f(y) - f(x)|\,dy < \delta$ whenever $0 < r < \eta$. In polar coordinates

$$G(r) = \int_0^r s^{n-1} g(s)\,ds < \nu_n \delta r^n, \quad r < \eta, \tag{2.8}$$

where $g(s) = \int\limits_{S^{n-1}} |f(x - s\,u) - f(x)|\, d\sigma_{n-1}(u)$. On the other hand, for all $\varepsilon > 0$ we have

$$\begin{aligned}|f * \varphi_\varepsilon(x) - a\, f(x)| \leq & \Big| \int_{|t|<\eta} [f(x-t) - f(x)]\varphi_\varepsilon(t)\, dt \Big| \\ & + \Big| \int_{|t|\geq\eta} [f(x-t) - f(x)]\varphi_\varepsilon(t)\, dt \Big| \\ & = I_1 + I_2.\end{aligned}$$

To do estimates on I_1 we make the following observations. First, note that ψ is a radial function, i.e. $\psi(x_1) = \psi(x_2)$ if $|x_1| = |x_2|$. Let us define $\psi_0(r) = \psi(x)$ where $r = |x|$. Clearly ψ_0 is a non increasing function. Using polar coordinates we get that

$$\begin{aligned}\int_{\mathbb{R}^n} \psi(x)\, dx \quad & \geq \quad \int_{r/2\leq|x|\leq r} \psi(x)\, dx \\ & = \quad \int_{r/2}^{r} \int_{S^{n-1}} \psi_0(s)s^{n-1}\sigma_{n-1}(du)\, ds \geq \nu_n \frac{2^n - 1}{2^n} r^n \psi_0(r).\end{aligned}$$

Hence $r^n\psi_0(r) \to 0$ as $r \to \infty$, and as $r \to 0$. Also, for some constant B_n depending on dimension n, $r^n\psi_0(r) \leq \|\psi\|_1 B_n$ for all r. Using these observations on equation 2.8 we get

$$\begin{aligned}I_1 \quad & \leq \quad \int\limits_{|y|<\eta} |f(x-y) - f(x)||\varphi_\varepsilon(t)|\, dt \leq \int\limits_0^\eta s^{n-1} g(s)\varepsilon^{-n}\psi_0(\varepsilon^{-1}s)\, ds \\ & = \quad G(s)\varepsilon^{-n}\psi_0(\varepsilon^{-1}s)\big]_0^\eta - \int\limits_0^\eta G(s)\varepsilon^{-n}\, d\psi_0(\varepsilon^{-1}s) \\ & \leq \quad \nu_n s^n \delta\varepsilon^{-n}\psi_0(\varepsilon^{-1}s)\big]_0^\eta - \int\limits_0^{\eta/\varepsilon} G(\varepsilon s)\varepsilon^{-n}\, d\psi_0(s) \\ & \leq \quad \nu_n\delta\|\psi\|_1 B_n - \nu_n\delta \int\limits_0^{\eta/\varepsilon} s^n\, d\psi_0(s) \leq \nu_n\delta\Big(\|\psi\|_1 B_n - \int_0^\infty s^n\, d\psi_0(s)\Big) \\ & = \quad \nu_n\delta\Big(\|\psi\|_1 B_n + n\int\limits_0^\infty s^{n-1}\psi_0(s)\, ds\Big) = \delta\Big(\nu_n B_n + 1\Big)\|\psi\|_1\end{aligned}$$

To estimate I_2, note that $\mathbb{1}_{B^c(0;\eta)}\psi \in \mathcal{L}_q$ for any $1 \leq q \leq \infty$. If $1/p + 1/p' = 1$, then by Hölder's inequality

$$I_2 \leq \|f\|_p \|\mathbb{1}_{B^c(0;\eta)}\psi_\varepsilon\|_{p'} + |f(x)|\, \|\|\mathbb{1}_{B^c(0;\eta)}\psi_\varepsilon\|_1$$

Since $\|\mathbb{1}_{B^c(0;\eta)}\psi_\varepsilon\|_1 = \int_{|x|\geq\eta}\psi_\varepsilon(x)\,dx = \int_{|x|\geq\eta/\varepsilon}\psi(x)\,dx$, the second term tends to 0 as $\varepsilon\to 0$. To estimate the first term, notice that $p' = 1 + p'/p$; hence,

$$\begin{aligned}\|\mathbb{1}_{B^c(0;\eta)}\psi_\varepsilon\|_{p'} &= \Big(\int_{|x|\geq\eta}[\psi_\varepsilon(x)]^{p'}\,dx\Big)^{\frac{1}{p'}} = \Big(\int_{|x|\geq\eta}\psi_\varepsilon(x)[\psi_\varepsilon(x)]^{\frac{p'}{p}}\,dx\Big)^{\frac{1}{p'}}\\ &\leq \Big(\|\mathbb{1}_{B^c(0;\eta)}\psi_\varepsilon\|_\infty^{\frac{p'}{p}}\int_{|x|\geq\eta}\psi_\varepsilon(x)\,dx\Big)^{\frac{1}{p'}}\\ &= \Big\|\mathbb{1}_{B^c(0;\eta)}\psi_\varepsilon\Big\|_\infty^{\frac{1}{p}}\Big\|\mathbb{1}_{B^c(0;\eta)}\psi_\varepsilon\Big\|_1^{\frac{1}{p'}}\end{aligned}$$

On the other hand $\|\mathbb{1}_{B^c(0;\eta)}\psi_\varepsilon\|_\infty = \varepsilon^{-n}\psi(\varepsilon^{-1}\eta) = \eta^n(\eta/\varepsilon)^n\psi_0(\eta/\varepsilon)\to 0$ as $\varepsilon\to 0$ as pointed out before, and (i) follows.
As for part (ii), note that $|(f*\varphi_\varepsilon)(x)| \leq (|f|*|\varphi_\varepsilon|)(x) \leq (|f|*\psi_\varepsilon)(x)$. Thus we only need to consider $f\geq 0$ and show that

$$\sup_{\varepsilon>0}|(f*\psi_\varepsilon)(x)| \leq \|\psi\|_1\, M_f(x)$$

Also, since

$$\begin{aligned}((\tau_{-x}f)*\psi_\varepsilon)(0) &= (f*\psi_\varepsilon)(x)\\ M_{\tau_{-x}f}(0) &= M_f(x)\\ (f*\psi_\varepsilon)(0) &= ((\delta_\varepsilon f)*\psi)(0)\end{aligned}$$

it suffices to prove that

$$(f*\psi)(0)\leq\|\psi\|_1 M_f(0)\quad\text{for all}\quad f\in\mathcal{L}_p$$

Let

$$\begin{aligned}\lambda(r) &= \int_{S^{n-1}} f(ru)\,\sigma_{n-1}(du)\\ \Lambda(r) &= \int_0^r \lambda(s)s^{n-1}\,ds = \int\limits_{|x|\leq r} f(x)\,dx\end{aligned}$$

$(f*\psi)(0) = \int f(-y)\psi(y)\,dy = \int f(y)\psi(-y)\,dy = \int f(y)\psi(y)\,dy$. Thus, using polar coordinates, the radial property of ψ and integration by parts we obtain

$$(f*\psi)(0) = \lim_{\substack{\varepsilon\to 0\\ N\to\infty}}\Big(\psi_0(r)\Lambda(r)\Big]_\varepsilon^N + \int\limits_\varepsilon^N \Lambda(r)\,d(-\psi_0(r))\Big)$$

For the first term converges to 0 as the following estimate shows it:

$$|\psi_0(N)\Lambda(N) - \psi_0(\varepsilon)\Lambda(\varepsilon)| \leq \nu_n M_f(0)(\psi_0(N)N^n + \psi_0(\varepsilon)\varepsilon^n)$$

For the second term we have the following estimate

$$\begin{aligned}\int_{\varepsilon}^{N} \Lambda(r)\, d(-\psi_0(r)) &\leq \nu_n M_f(0) \int_{0}^{\infty} r^n\, d(-\psi_0(r)) \\ &= \|\psi\|_1 M_f(0)\end{aligned}$$

This concludes our proof. □

Any point of continuity of f is of course a Lebesgue point of f. Thus if f is continuous at 0, then by theorems (2.11) and (2.18) we have that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \hat{f}(x) e^{-2\pi\varepsilon|x|}\, dx = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} f(x) P(x,\varepsilon)\, dx = f(0)$$

If we further assume that $\hat{f} \geq 0$ it follows from monotone convergence that $\hat{f} \in \mathcal{L}_1(\mathbb{R}^n)$. In this way we obtain the following simple and neat result:

Corollary 2.19. *Suppose* $f \in \mathcal{L}_1(\mathbb{R}^n)$ *and* $\hat{f} \geq 0$. *If* f *is continuous at* 0 *then* $\hat{f} \in \mathcal{L}_1(\mathbb{R}^n)$ *and*

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(t) e^{2\pi i t\cdot x}\, dt$$

almost everywhere (i.e. at every Lebesgue point of f*). In particular,*

$$f(0) = \int_{\mathbb{R}^n} \hat{f}(t)\, dt$$

Applying this result to the Poisson and Gauss–Weierstrass kernels we get:

Corollary 2.20.

$$\begin{aligned}\int_{\mathbb{R}^n} W(x,\alpha) e^{2\pi i t\cdot x}\, dx &= e^{-4\pi^2\alpha|t|^2} \\ \int_{\mathbb{R}^n} P(x,\alpha) e^{2\pi i t\cdot x}\, dx &= e^{-2\pi\alpha|t|}\end{aligned}$$

for all $\alpha > 0$.

We immediately obtain the *semigroup* properties of the Poisson and Gauss–Weierstrass kernels:

Corollary 2.21. *If* α_1 *and* α_2 *are positive real numbers then*

$$\begin{aligned}\text{(a)}\quad W(x,\alpha_1+\alpha_2) &= \int_{\mathbb{R}^n} W(x-t,\alpha_1) W(t,\alpha_2)\, dt \\ \text{(b)}\quad P(x,\alpha_1+\alpha_2) &= \int_{\mathbb{R}^n} P(x-t,\alpha_1) P(t,\alpha_2)\, dt\end{aligned}$$

3. $\mathcal{L}_2(\mathbb{R}^n)$ Theory and the Plancherel Theorem

We know that the space $\mathcal{L}_1(\mathbb{R}^n) \cap \mathcal{L}_2(\mathbb{R}^n)$ is dense in $\mathcal{L}_2(\mathbb{R}^n)$. Here we will extend the Fourier transform from the former space to the latter. One of the nice properties of this extension is that it turns out to be a unitary linear transformation.

Theorem 3.1. *If $f \in \mathcal{L}_1(\mathbb{R}^n) \cap \mathcal{L}_2(\mathbb{R}^n)$ then $\|\hat{f}\|_2 = \|f\|_2$*

Proof. Let $g(x) = \overline{f(-x)}$. Then by theorem 2.3 $h = f * g$ is continuous. Being the convolution of functions in $\mathcal{L}_1(\mathbb{R}^n)$ h is also integrable. Since $\hat{g} = \overline{\hat{f}}$ then $\hat{h} = \hat{f}\hat{g} = |\hat{f}|^2$. By corollary 2.19 it follows that $\hat{h}$ is integrable and $h(0) = \int_{\mathbb{R}^n} \hat{h}(t)\,dt$. Thus

$$\begin{aligned} \int_{\mathbb{R}^n} |\hat{f}(t)|^2 dt &= \int_{\mathbb{R}^n} \hat{h}(t)\,dt = h(0) = \int_{\mathbb{R}^n} f(y)g(0-y)\,dy \\ &= \int_{\mathbb{R}^n} f(y)\overline{f(y)}\,dy = \int_{\mathbb{R}^n} |f(y)|^2\,dy \end{aligned}$$

□

This theorem asserts that the Fourier transform is a unitary operator defined on the dense subspace $L_1 \cap L_2$ of $\mathcal{L}_2(\mathbb{R}^n)$ into $\mathcal{L}_2(\mathbb{R}^n)$. Thus by Caratheódory extension, there exists a unique bounded extension, $\mathcal{F}$ on the whole space $\mathcal{L}_2(\mathbb{R}^n)$. We will also use the notation $\hat{f} = \mathcal{F}f$. If $f \in \mathcal{L}_2$, then $\mathcal{L}_1 \cap \mathcal{L}_2 \ni h_k = f\mathbb{1}_{B(0;k)}$ converges in L_2 to f; thus, $\hat{h}_k \in \mathcal{L}_2$ and

$$\hat{f}(t) = \mathcal{F}f(t) = \lim_{k\to\infty} \hat{h}_k(t) = \lim_{k\to\infty} \int_{|x|\le k} f(x)e^{-2\pi i x\cdot t}\,dx$$

in $\mathcal{L}_2$. It is also clear that $\mathcal{F}$ is a unitary operator from $\mathcal{L}_2(\mathbb{R}^n)$ into $\mathcal{L}_2(\mathbb{R}^n)$. In fact this operator is onto:

Theorem 3.2. (Plancherel)

(i) *The Fourier transform is a unitary operator on $\mathcal{L}_2(\mathbb{R}^n)$.*
(ii) *The inverse Fourier transform, $\mathcal{F}^{-1}$, can be obtained by letting $(\mathcal{F}^{-1}g)(x) = (\mathcal{F}\hat{g})(-x)$ for all $g \in \mathcal{L}_2(\mathbb{R}^n)$.*

Proof. Since $\mathcal{F}$ is a unitary operator, $\mathcal{F}(\mathcal{L}_2(\mathbb{R}^n))$ is a closed subspace of $\mathcal{L}_2(\mathbb{R}^n)$. Let $g \in \Big(\mathcal{F}(\mathcal{L}_2(\mathbb{R}^n))\Big)^{\perp}$. A simple density argument extends theorem 2.10 to functions in $\mathcal{L}_2(\mathbb{R}^n)$. Thus $\int_{\mathbb{R}^n} \hat{f}(t)g(x)\,dx = \int_{\mathbb{R}^n} f(x)\hat{g}(x)\,dx = 0$ for all $f \in \mathcal{L}_2(\mathbb{R}^n)$. Therefore $\|g\|_2 = \|\hat{g}\|_2 = 0$.
For the second part of the proof, note that since $\mathcal{F}$ is a unitary operator, then it preserves the inner product: $(u|v) = \int_{\mathbb{R}^n} u\overline{v}\,dx$. Let g be any function in $L_1 \cap L_2$ and $f \in \mathcal{L}_2$. Since $\hat{f} \in \mathcal{L}_2$, then $\hat{f}\mathbb{1}_{B(0;k)} \in \mathcal{L}_1 \cap \mathcal{L}_2$ for each k. Hence

$$f_k(x) = \int_{|t|\le k} \hat{f}(t)e^{2\pi i x\cdot t}\,dt = \mathcal{F}(\hat{f}\mathbb{1}_{B(0;k)})(-x) \in \mathcal{L}_2$$

converges to the function $\tilde{f}(x) = \mathcal{F}(\hat{f})(-x)$ in $\mathcal{L}_2$. Consequently,

$$\begin{aligned}(g|\tilde{f}) = \lim_{k\to\infty}(g|f_k) &= \lim_{k\to\infty}\int_{\mathbb{R}^n} g(x)\Big(\int_{|t|\le k}\overline{\hat{f}(t)e^{-2\pi i t\cdot x}\,dt}\Big)\,dx \\ &= \lim_{k\to\infty}\int_{|t|\le\ k}\overline{\hat{f}(t)}\Big(\int_{\mathbb{R}^n} g(x)e^{-2\pi i t\cdot x}\,dx\Big)\,dt \\ &= \lim_{k\to\infty}\int_{|t|\le k}\hat{g}(t)\overline{\hat{f}(t)}\,dt = (\hat{g}|\hat{f}) = (g|f)\end{aligned}$$

This clearly implies that $f(x) = (\mathcal{F}^{-1}\hat{f})(x) = \tilde{f}(x) = (\mathcal{F}\hat{f})(-x)$ for all $f \in \mathcal{L}_2(\mathbb{R}^n)$. □

4. Tempered Distributions and a Characterization of Operators that Commute with Translations

The basic idea in the theory of distributions is to consider them as linear functionals on a space of *regular functions* –often called *testing functions*. This space is assumed to be well behave with respect to operations we have studied so far, namely: differentiation, Fourier transform, convolution, translation, dilation, etc. This will be reflected, in return, on the properties of distributions. We are after a space of testing functions $\mathcal{S}$ on which these operations are defined and preserve the space. Motivated by theorems 2.5, 2.6 and formulas 2.1, the space $\mathcal{S}$ must consist of infinitely differentiable functions such that when they or their derivatives are multiplied by polynomials, they must still be integrable.

Definition 4.1. The space $\mathcal{S}$ of *testing functions* is defined to be the class of all functions $\phi \in \mathcal{C}^\infty$ on $\mathbb{R}^n$ such that

$$\sup_{x\in\mathbb{R}^n} |x^\alpha (D^\beta\phi)(x)| < \infty$$

for all n–tuples $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ of nonnegative integers.

Clearly that space is not void: the family of functions $\varphi_\alpha(x) = e^{-\alpha|x|^2}$ belongs to $\mathcal{S}$. $\mathcal{S}$ also contains the space $\mathcal{D}$ of all $\mathcal{C}^\infty$ functions with compact support. The later space is not void. To see this, we first consider the case $n = 1$. Let $f(t) = e^{-1/t}$ if $t > 0$ and $f(t) = 0$ when $t \le 0$. It is an easy exercise to see that $f \in \mathcal{C}^\infty$ and that f and all its derivatives are bounded. Let $\varphi(t) = f(1+t)f(1-t)$; then $\varphi(t) = e^{-2/(1-|t|^2)}$ if $|t| < 1$ and zero otherwise. Thus $\varphi \in \mathcal{D}(\mathbb{R})$. We can easily obtain n–dimensional variants from φ:

(a) For $x \in \mathbb{R}^n$ define $\psi(x) = \varphi(x_1)\cdots\varphi(x_n)$; then $\psi \in \mathcal{D}(\mathbb{R}^n)$
(b) For $x \in \mathbb{R}^n$ define $\psi(x) = e^{-2/(1-|x|^2)}$, if $|x| < 1$ and zero otherwise. Then $\psi \in \mathcal{D}(\mathbb{R}^n)$
(c) If $\eta \in \mathcal{C}^\infty$ and ψ is the function in (b) then $\psi(\varepsilon x)\eta(x)$ defines a function in $\mathcal{D}(\mathbb{R}^n)$.

The spaces $\mathcal{C}_0$ (continuous functions that vanish at infinity with the sup norm) and $\mathcal{L}_p(\mathbb{R}^n)$, with $1 \leq p < \infty$ contain $\mathcal{S}$. Using the density of $\mathcal{C}_c$ on both spaces and the Stone–Weierstrass theorem we can show that $\mathcal{D}$ is in fact dense in both spaces and therefore so would $\mathcal{S}$. Another interesting observation is given by the following

Lemma 4.1. *(a) If $\varphi \in \mathcal{S}$, then φ is $\mathcal{L}_p$ differentiable for any $1 \leq p \leq \infty$. (b) The L^p norm of a function $\varphi \in \mathcal{S}$ is bounded by a linear combination of L_∞ norms of the form $x^\alpha \varphi(x)$; the coefficients and exponents of these terms depend only on p and the dimension n.*

Proof. (a) It is enough to consider $\partial_{x_1}\psi$ for $\psi \in \mathcal{S}$. There is a constant $A > 0$ such that $|\partial_{x_1}\psi(x)| \leq A/|x|^{2n+1}$. Consider

$$g = \sum_{k=0}^{\infty} \sup_{\{k \leq |x| < k+1\}} |\partial_{x_1}\psi|\, \mathbb{1}_{\{k \leq |x| < k+1\}} \leq \sum_{k=0}^{\infty} \frac{1}{k^{2n+1}}\, \mathbb{1}_{\{k \leq |x| < k+1\}} \in \mathcal{L}_p$$

The mean value theorem implies

$$\Big|\frac{\psi(x+h\mathbf{e}_1) - \psi(x)}{h}\Big| = |\partial_{x_1}\psi(\xi_{x,h})| \leq g(x)$$

where $\xi_{x,h}$ is in the line between x and $x+h\mathbf{e}_1$. The conclusion follows from dominated convergence.
(b) Let $A = \|\varphi\|_\infty$ and $B = \sup_{x\in\mathbb{R}^n} |x|^{2n}|\varphi(x)|$ then

$$\begin{aligned}
\Big(\int_{\mathbb{R}^n} |\varphi(x)|^p\,dx\Big)^{1/p} &\leq \Big(\int_{|x|\leq 1} |\varphi(x)|^p\,dx\Big)^{1/p} + \Big(\int_{|x|>1} |\varphi(x)|^p\,dx\Big)^{1/p} \\
&\leq A\Big(\frac{\sigma_{n-1}}{n}\Big)^{1/p} + B\Big(\int_{|x|>1} |x|^{-2np}\,dx\Big)^{1/p} \\
&= A\Big(\frac{\sigma_{n-1}}{n}\Big)^{1/p} + B\Big(\frac{\sigma_{n-1}}{(2p-1)n}\Big)^{1/p}
\end{aligned}$$

□

Corollary 4.2. *If $\varphi \in \mathcal{S}$ then $\hat{\varphi} \in \mathcal{S}$.*

Proof. Theorem 2.5 implies that $\hat{\varphi} \in \mathcal{C}^\infty$ whenever $\varphi \in \mathcal{S}$. Combining this with Theorems 2.6 and 4.1(a) we conclude that $\hat{\varphi} \in \mathcal{S}$. □

Remember that $(\varphi * \psi)^\wedge = \hat{\varphi}\hat{\psi}$. Since $\mathcal{S}$ is closed under multiplication, then the inverse Fourier transform theorem shows that

Theorem 4.3. *If φ and ψ are in $\mathcal{S}$ then so is $\varphi * \psi$.*

We introduce now a metric on $\mathcal{S}$ that makes this space a topological vector space. There is a natural countable family of norms $\{\rho_{\alpha\beta}\}$ in $\mathcal{S}$, indexed by ordered pairs (α, β) of n-tuples of nonnegative integers, given by:

$$\rho_{\alpha\beta}(\varphi) = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta \varphi(x)|$$

These norms induce the distances $d_{\alpha\beta}(\varphi,\psi) = \rho_{\alpha\beta}(\varphi-\psi)$. Let $\{d_k\}$ be an enummeration of these metrics. Thus, we define a metric d in $\mathcal{S}$ as

$$d(\varphi,\psi) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_n(\varphi,\psi)}{1+d_n(\varphi,\psi)}$$

It is clear that $\varphi_m \to \varphi$ with respect to d, if and only if $\varphi_m \to \varphi$ with respect to each d_k (as $m \to \infty$). From this observation, it follows that the space operations $(\varphi,\psi) \mapsto \varphi+\psi$ and $(a,\varphi) \mapsto a\varphi$ are continuous. Thus $\mathcal{S}$ with the topology induced by d becomes a topological vector space. The following properties of $\mathcal{S}$ are not difficult to check.

Theorem 4.4. $(\mathcal{S}, d)$ *has the following properties:*

1. *The mapping* $\varphi(x) \mapsto x^\alpha D^\beta \varphi(x)$ *is continuous*
2. *If* $\varphi \in \mathcal{S}$ *then* $\lim_{h\to 0} \tau_h \varphi = \varphi$
3. *Suppose* $\varphi \in \mathcal{S}$ *and* $h = (0,\dots,h_i,\dots,0)$ *lies on the i-th coordinate axis of* $\mathbb{R}^n$*, then the difference quotient* $[\varphi - \tau_h\varphi]/h_i$ *tend to* $\partial\varphi/\partial x_i$ *as* $h \to 0$
4. $\mathcal{S}$ *is a complete metric space*
5. *The Fourier transform is a homeomorphism of* $\mathcal{S}$ *onto itself.*
6. $\mathcal{D}$ *is dense subset of* $\mathcal{S}$
7. $\mathcal{S}$ *is separable.*

The collection $\mathcal{S}'$ of all continuous linear functionals on $\mathcal{S}$ is called *the space of tempered distributions*. Thus to show that a linear functional L on $\mathcal{S}$ is a tempered distribution it suffices to show continuity at the origin, i.e, $\lim_{k\to\infty} L(\varphi_k) = 0$ whenever $\varphi_k \to 0$ in $\mathcal{S}$. Here we present some examples:

Example 4 Let $f \in \mathcal{L}_p(\mathbb{R}^n)$, $1 \le p \le \infty$, and let $L = L_f$ defined as

$$L(\varphi) = L_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx$$

for $\varphi \in \mathcal{S}$. Lemma 4.1 implies that $\|\varphi_k\|_q \to 0$ if $\varphi_k \to 0$ in $\mathcal{S}$. By Hölder's inequality

$$|L(\varphi)| \le \|f\|_p \|\varphi\|_q$$

Thus, L is a tempered distribution

Example 5 If μ is a finite Borel measure, then the linear functional $L = L_\mu$ defined by

$$L(\varphi) = L_\mu(\varphi) = \int_{\mathbb{R}^n} \varphi(x)\,d\mu(x)$$

is a tempered distribution.

Example 6 Let f be a measurable function such that $f(x)(1+|x|^2)^{-k}$ belongs to $\mathcal{L}_p(\mathbb{R}^n)$, $1 \le p \le \infty$ for some integer k. Define $L = L_f$ just as in example 4: $L(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx$. Since

$$L(\varphi) = \int_{\mathbb{R}^n} \left[(1+|x|^2)^k \varphi(x)\right] \left[f(x)/(1+|x|^2)^{-k}\right]\,dx$$

it follows that L is a tempered distribution. f is called a *tempered $\mathcal{L}_p$ function*. When $p = \infty$, f is said to be *slowly increasing*.

Example 7 A Borel measure μ is called a *tempered measure* if

$$\int_{\mathbb{R}^n} (1 + |x|^2)^{-k} \, d|\mu|(x) < \infty$$

for some integer k. If we define $L(\varphi) = L_\mu(\varphi) = \int_{\mathbb{R}^n} \varphi(x) \, d\mu(x)$, then we get a tempered distribution.

Example 8 For a fixed point $x_0 \in \mathbb{R}^n$, and an n-tuple β of nonnegative integers, we define the linear functional $L(\varphi) = D^\beta \varphi(x_0)$. The continuity of the metric $\rho_{0\beta}$ in $\mathcal{S}$ implies that L is a tempered distribution. In particular, for $x_0 = 0$ and $\beta = (0, \ldots, 0)$ we get the *Dirac's delta distribution*: $L(\varphi) = \varphi(0)$. The latter can be obtained also, by considering the Borel measure of mass 1 concentrated at the origin.

There is a simple characterization of tempered distributions.

Theorem 4.5. *A linear functional L on $\mathcal{S}$ is a tempered distribution if and only if there exists a constant $C > 0$ and integers m and l such that*

$$|L(\varphi)| \leq C \sum_{|\alpha| \leq l, |\beta| \leq m} \rho_{\alpha\beta}(\varphi)$$

for all $\varphi \in \mathcal{S}$.

Proof. The existence of C, l, m clearly implies the continuity of L.

For the converse, note that the family of sets of the form

$$N_{\varepsilon,l,m} = \Big(\varphi : \sum_{|\alpha| \leq l, |\beta| \leq m} \rho_{\alpha\beta}(\varphi) < \varepsilon\Big)$$

where $\varepsilon > 0$, and l and m are nonnegative integers, forms a basis of neighborhoods around 0 in $\mathcal{S}$. To see that, let $k = lm$ and consider the sets

$$U_{\alpha\beta,\varepsilon} = \Big(\varphi : \rho_{\alpha\beta}(\varphi) < \varepsilon/k\Big)$$

Clearly, these are open sets in $\mathcal{S}$. Also

$$\bigcap_{|\alpha| \leq l, |\beta| \leq m} U_{\alpha\beta\varepsilon} \subset N_{\varepsilon,l,m}.$$

For the ball $B(0, \varepsilon)$ in $\mathcal{S}$, pick $K > 0$ such that $\sum_{n=K}^{\infty} 2^{-n} < \varepsilon/2$. Take l and m big enough such that the norms in the list $\{\rho_1, \ldots, \rho_K\}$ are included among the norms $\{\rho_{\alpha\beta}\}$ with $|\alpha| \leq l, |\beta| \leq m$. Then

$$N_{\varepsilon/2,l,m} \subset \bigcap_{n=1}^{K} \{\varphi : \rho_n(\varphi) < \varepsilon/2\} \subset B(0, \varepsilon)$$

It follows that there is a set $N_{\varepsilon,l,m}$ such that $|L(\varphi)| < 1$ for all $\varphi \in N_{\varepsilon,l,m}$. For $\varphi \neq 0$, let

$$\|\varphi\| = \sum_{|\alpha|\leq l, |\beta|\leq m} \rho_{\alpha\beta}(\varphi)$$

it is easy to see now that $C = 2/\varepsilon$ will do the job:

$$|L(\varphi)| = \frac{2\|\varphi\|}{\varepsilon}\left|L\Big(\frac{\varepsilon}{2\|\varphi\|}\varphi\Big)\right| \leq \frac{2\|\varphi\|}{\varepsilon}$$

□

Many important operations on functions, i.e: differentiation, translation, dilation, reflections, Fourier transform, convolution, etc, can be extended in a natural way to distributions. Here we motivate some of these extension.

Derivative:

If $f \in \mathcal{L}_p(\mathbb{R}^n)$ admits an $\mathcal{L}_p(\mathbb{R}^n)$ derivative $g = D^\alpha f$, then by integration by parts we have that

$$L_g(\varphi) = \int_{\mathbb{R}^n} (D^\alpha f(x))\varphi(x)\,dx = (-1)^{|\alpha|}\int_{\mathbb{R}^n} f(x)D^\alpha\varphi(x)\,dx = (-1)^{|\alpha|}L_f(D^\alpha\varphi)$$

for all $\varphi \in \mathcal{S}$. Thus, for any distribution u, we define $D^\alpha u$ as the linear functional$v$ on $\mathcal{S}$ such that

$$v(\varphi) = (-1)^{|\alpha|}u(D^\alpha\varphi)$$

Clearly $D^\alpha u$ is a distribution.

Translation:

If $f \in \mathcal{L}_p(\mathbb{R}^n)$ then

$$L_{\tau_h f}(\varphi) = \int_{\mathbb{R}^n} f(x-h)\varphi(x)\,dx = \int_{\mathbb{R}^n} f(x)\varphi(x+h)\,dx = L_f(\tau_{-h}\varphi)$$

Thus, for any distribution u, we define $\tau_h u$ as the linear functional on $\mathcal{S}$ such that

$$\tau_h u(\varphi) = u(\tau_{-h}\varphi)$$

It is clear that $\tau_h u$ is a distribution since it is the composition of continuous functions on $\mathcal{S}$.

Fourier transform:

If $f \in \mathcal{L}_1(\mathbb{R}^n)$ then from theorem 2.10 we have

$$L_{\hat{f}}(\varphi) = \int_{\mathbb{R}^n} \hat{f}(x)\varphi(x)\,dx = \int_{\mathbb{R}^n} f(x)\hat{\varphi}(x)\,dx = L_f(\hat{\varphi})$$

thus, for any given distribution u we define $\hat{u}$ by

$$\hat{u}(\varphi) = u(\hat{\varphi})$$

Multiplication by a test function:

If $\varphi \in \mathcal{S}$ and $u \in \mathcal{S}'$ then we define φu as the linear functional

$$(\varphi u)(\psi) = u(\varphi\psi)$$

for all $\psi \in \mathcal{S}$. Clearly this is a distribution on $\mathcal{S}$.
Convolution:
If $f \in \mathcal{L}_p(\mathbb{R}^n)$ and φ and ψ belong to $\mathcal{S}$. Let $\tilde{\varphi}(x) = \varphi(-x)$. A simple application of Fubini's theorem shows that

$$L_{f*\varphi}(\psi) = \int_{\mathbb{R}^n} (f*\varphi)(x)\psi(x)\,dx = \int_{\mathbb{R}^n} f(x)(\tilde{\varphi}*\psi)(x)\,dx = L_f(\tilde{\varphi}*\psi)$$

Thus, for any distribution u and test function φ we define $u*\varphi$ as the linear functional

$$(u*\varphi)(\psi) = u(\tilde{\varphi}*\psi)$$

The right side of the equation above is the composition of continuous functions in $\mathcal{S}$. Therefore $u*\varphi$ is a distribution.
The following theorem gives a better characterization of the convolution distribution just described:

Theorem 4.6. *If $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$ then the convolution $u*\varphi$ is the function $f(x) = u(\tau_x\tilde{\varphi})$. Moreover, $f \in \mathcal{C}^\infty$ and it is slowly increasing.*

Proof. First we show that $(u*\varphi)(\psi) = \int_{\mathbb{R}^n} f(t)\psi(t)\,dt$:

$$\begin{aligned}(u*\varphi)(\psi) = u(\tilde{\varphi}*\psi) &= u\Big(\int_{\mathbb{R}^n} \tilde{\varphi}(x-t)\psi(t)\,dt\Big)\\ &= u\Big(\int_{\mathbb{R}^n} (\tau_t\tilde{\varphi})(x)\psi(t)\,dt\Big)\end{aligned}$$

The Riemann sums of the last integral converge in the topology of $\mathcal{S}$, thus

$$u\Big(\int_{\mathbb{R}^n} (\tau_t\tilde{\varphi})(x)\psi(t)\,dt\Big) = \int_{\mathbb{R}^n} u(\tau_t\tilde{\varphi})\psi(t)\,dt = \int_{\mathbb{R}^n} f(t)\psi(t)\,dt$$

Continuity follows from theorem 4.4. Let $h = (0,\dots,h_j,\dots,0)$, then

$$\frac{\tau_{x+h}\tilde{\varphi} - \tau_x\tilde{\varphi}}{h_j} \longrightarrow -\tau_x\Big(\frac{\partial\tilde{\varphi}}{\partial x_j}\Big)$$

in the $\mathcal{S}$ topology. Thus, by continuity of u,

$$\frac{f(x+h)-f(x)}{h_j} \rightarrow u\Big(-\tau_x\Big(\frac{\partial\tilde{\varphi}}{\partial x_j}\Big)\Big)$$

By an iterative argument we have $(D^\alpha f)(x) = (-1)^{|\alpha|}u(\tau_x D^\alpha\tilde{\varphi})$. The fact that f is slowly increasing follows from theorem 4.5: There exist $C > 0$ and nonnegative integers l and m such that

$$|f(x)| = u(\tau_x\tilde{\varphi}) \le C \sum_{|\alpha|\le l, |\beta|\le m} \rho_{\alpha\beta}(\tau_x\tilde{\varphi})$$

But

$$\rho_{\alpha\beta}(\tau_x\tilde{\varphi}) = \sup_{w\in\mathbb{R}^n} |(w+x)^\alpha(D^\beta\tilde{\varphi})(w)|$$

which is clearly a polynomial in x. □

In what follows, we will apply the theory of distributions to the study of a class of operators that occur in the theory of singular integrals: those that commute with translations. Suppose B is an operator mapping a linear space V of functions in $\mathbb{R}^n$ into another space of the like. We say that B *commutes with translations* if $\tau_h B = b\tau_h$ for all $h \in \mathbb{R}^n$.

Example 9:
Fix $f \in \mathcal{L}_p(\mathbb{R}^n)$ and let be B the operator on $\mathcal{L}_1(\mathbb{R}^n)$ to $\mathcal{L}_p(\mathbb{R}^n)$ defined by convolution: $g \mapsto f * g$. We have that $\|Bg\|_p \leq \|f\|_p \|g\|_1$. Clearly it commutes with translations.

Example 10:
Consider the Banach space of all complex measures on $\mathbb{R}^n$ with norm $\|\mu\|$. The Fourier transform of f as a distribution coincides with that of a measure:

$$\hat{\mu}(t) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot t} \, d\mu(x)$$

Let ξ be the map:

$$\begin{aligned} \xi : \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (x, y) &\mapsto x + y \end{aligned}$$

Let $\mu * \lambda$ be the measure on $\mathbb{R}^n$ induced by ξ, i.e:

$$(\mu * \lambda)(E) = (\mu \times \lambda)\Big(\xi^{-1}(E)\Big)$$

It follows straight forward that

$$\|\mu * \lambda\| \leq \|\mu\| \|\lambda\|$$

From the Riesz–representation theorem, it follows that $\mu * \lambda$ is the unique measure ν such that

$$\int_{\mathbb{R}^n} f(x) \, d\nu(x) = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x + y) \, d(\mu \otimes \lambda)(x, y)$$

for every $f \in \mathcal{C}_c(\mathbb{R}^n)$. Application of Fubuni's theorem implies

$$(\mu * \lambda)(E) = \int_{\mathbb{R}^n} \mu(E - t) \, d\lambda(t) = \int_{\mathbb{R}^n} \mu(E - s) \, d\mu(s)$$

and if $d\lambda = f\,dx$, where $f \in L^1(\mathbb{R}^n)$ then $d(\mu * \lambda)(y) = (f * \mu)(y)\,dy$:

$$\begin{aligned}(\mu * \lambda)(E) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbf{1}_E(x+y)\,d(\mu \otimes \lambda)(x,y) \\ &= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \mathbf{1}_E(x+y) f(y)\,dy \Big)\,d\mu(x) \\ &= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \mathbf{1}_E(y) f(y-x)\,dy \Big)\,d\mu(x) \\ &= \int_{E} \Big(\int_{\mathbb{R}^n} f(y-x)\,d\mu(x) \Big)\,dy = \int_E (f * \mu)(y)\,dy\end{aligned}$$

It is clear that the map on $\mathcal{L}_1(\mathbb{R}^n)$ given by $f \mapsto f * \mu$ is continuous: $\|f * \mu\|_1 = \|\mu * \lambda\| \leq \|f\|_1 \|\mu\| = \|\mu\| \|\lambda\|$. Clearly it also commutes with translation.
We will show that all bounded operators in $\mathcal{L}_p(\mathbb{R}^n)$ are of the similar *convolution type*. The following lemma will imply the former statement:

Lemma 4.7. *If $f \in \mathcal{L}_p(\mathbb{R}^n)$ has derivatives in the L^p norm of all orders $\leq n+1$, then f equals almost everywhere a continuous function g satisfying*

$$|g(0)| \leq C_{n,p} \sum_{|\alpha| \leq n+1} \|D^\alpha f\|_p$$

where $C_{n,p}$ depends only on the dimension n and the exponent p.

Proof. Let $x \in \mathbb{R}^n$, then

$$(1 + |x|)^{(n+1)/2} \leq \Big(1 + \sum_{j=1}^{n} |x_j|\Big)^{n+1} \leq C' \sum_{|\alpha| \leq n+1} |x^\alpha|$$

For $p = 1$ we will prove that $\hat{f}$ is integrable and use corollary 2.16:

$$\begin{aligned}|\hat{f}(t)| &\leq C'(1 + |t|^2)^{-(n+1)/2} \sum_{|\alpha| \leq n+1} |t|^\alpha |\hat{f}(t)| \\ &= C'(1 + |t|^2)^{-(n+1)/2} \sum_{|\alpha| \leq n+1} |((2\pi)^{-|\alpha|} D^\alpha f)^\wedge(t)| \\ &\leq C''(1 + |t|^2)^{-(n+1)/2} \sum_{|\alpha| \leq n+1} \|D^\alpha f\|_1\end{aligned}$$

Therefore

$$\|\hat{f}\|_1 \leq \frac{C'' \sigma_n}{2} \sum_{|\alpha| \leq n+1} \|D^\alpha f\|_1$$

thus f is equal to a continuous function g almost everywhere and

$$|g(0)| \leq \|g\|_\infty = \|f\|_\infty \leq \|\hat{f}\|_1 \leq \frac{C'' \sigma_n}{2} \sum_{|\alpha| \leq n+1} \|D^\alpha f\|_1$$

For $p > 1$, choose $0 \leq \varphi \leq 1$ in $\mathcal{D}$ such that $\varphi(x) = 1$ if $|x| \leq 1$ and $\varphi(x) = 0$ if $|x| \geq 2$. Then φf satisfies the hypothesis of $p = 1$. Thus $varphif$ equals a continuous function h such that

$$|h(0)| \leq C \sum_{|\alpha| \leq n+1} \|D^\alpha \varphi f\|_1$$

Since $D^\alpha(\varphi f) = \sum_{\mu+\nu=\alpha} C^{\mu\nu}(D^\mu f)(D^\nu \varphi)$ and using Hölder's inequality we have

$$\begin{aligned}
\|D^\alpha(\varphi f)\|_1 &\leq \int_{|x|\leq 2} \sum_{\mu+\nu=\alpha} C^{\mu\nu} |D^\nu \varphi|\, |D^\mu f|\, dx \\
&\leq K \sum_{\mu+\nu=\alpha} \rho_{0\nu}(\varphi) \int_{|x|\leq 2} |D^\mu f|\, dx \\
&\leq K \Big(\frac{2^n \sigma_{n-1}}{n}\Big)^{1/q} \sum_{\mu+\nu=\alpha} \rho_{0\nu}(\varphi) \|D^\mu f\|_p \\
&\leq K \Big(\frac{2^n \sigma_{n-1}}{n}\Big)^{1/q} \max_{|\nu|\leq|\alpha|} \{\rho_{0\nu}(\varphi)\} \sum_{|\mu|\leq|\alpha|} \|D^\mu f\|_p
\end{aligned}$$

Thus, there is a constant $C_{n,p}$ depending only on n and p such that

$$h(0) \leq C_{n,p} \sum_{|\alpha|\leq n+1} \|D^\alpha f\|_p \tag{4.1}$$

Since $\varphi(x) = 1$ on the unit ball we see that f is equal to a continuous function h almost everywhere on the unit ball. Choosing $\varphi_k(x) = \varphi(x/k)$ shows that f is almost everywhere a continuous function h_k on the ball of radius k around 0. Note that $h_k(0) = h(0)$, hence equation (4.1) is preserved. This finishes the proof of the lemma. □

The following theorem characterizes all continuous operators on $\mathcal{L}_p(\mathbb{R}^n)$ that commute with translations:

Theorem 4.8. *Suppose $B : \mathcal{L}_p(\mathbb{R}^n) \longrightarrow \mathcal{L}_q(\mathbb{R}^n)$, $1 \leq p, q \leq \infty$, is a linear bounded operator that commutes with translations; then, there exists a unique tempered distribution u such that $B\varphi = u * \varphi$ for all $\varphi \in \mathcal{S}$.*

Proof. Let $h = (0, \ldots, h_j, \ldots, 0)$ be a nonzero vector along the j–th axis. Since $\frac{\tau_h\varphi - \varphi}{h_j} \to -\frac{\partial\varphi}{\partial x_j}$ in $\mathcal{S}$, it converges in L^p as well. Therefore, $B\Big(\frac{\tau_h\varphi-\varphi}{h_j}\Big) = \frac{\tau_h B\varphi - B\varphi}{h_j}$ converges to $-B\Big(\frac{\partial\varphi}{\partial x_j}\Big) = -\frac{\partial B\varphi}{\partial x_j}$ in L^q by continuity of B. By an iteration argument, it follows that $B\varphi$ has derivatives in L^q of all orders and $D^\alpha(B\varphi) = B(D^\alpha\varphi)$. Hence by the previous lemma, $B\varphi$ equals a continuous function g_φ almost everywhere and

$$\begin{aligned} |g_\varphi(0)| &\leq C_{n,q} \sum_{|\alpha|\leq n+1} \|D^\alpha(B\varphi)\|_q \\ &= C_{n,q} \sum_{|\alpha|\leq n+1} \|B(D^\alpha\varphi)\|_q \\ &\leq \|B\| C_{n,q} \sum_{|\alpha|\leq n+1} \|D^\alpha\varphi\|_q \end{aligned}$$

This shows that the linear functional $\varphi \mapsto g_\varphi(0)$ is in fact a distribution u_1 on $\mathcal{S}$. Letting $u = \tilde{u_1}$ we obtain the distribution we are looking for: $(u * \varphi)(x) = u(\tau_x\tilde{\varphi}) = u((\tau_{-x}\varphi^{\sim})) = \tilde{u}(\tau_{-x}\varphi) = u_1(\tau_{-x}\varphi) = (B(\tau_{-x}\varphi))(0) = (\tau_{-x}B\varphi)(0) = (B\varphi)(x)$. Uniqueness follows immediately from the construction. □

Let us denote by (L^p, L^q) the space of those tempered distributions u for which there exists $A > 0$ such that $\|u * \varphi\|_q \leq A\|\varphi\|_p$ for all $\varphi \in \mathcal{S}$. Last theorem shows that there is a one–to–one relationship between this space, and that of continuous linear operators from $\mathcal{L}_p(\mathbb{R}^n)$ to $\mathcal{L}_q(\mathbb{R}^n)$ that commute with translations. For the case $p = 2 = q$ there is a much better characterization:

Theorem 4.9. *The distribution u belongs to (L^2, L^2) if and only if there exists $b \in L^\infty(\mathbb{R}^n)$ such that $\hat{u} = b$. In this case, $\|b\|_\infty$ is the norm of the operator $B : L^2 \cap \mathcal{S} \to L^2$ defined by putting $B\varphi = u * \varphi$; moreover $(u * \varphi)^\wedge = \hat{u}\hat{\varphi}$.*

Proof. By the Fourier inverse theorem, we have that $\tilde{\varphi} * \hat{\psi}$ is the Fourier transform of $\hat{\varphi}\psi$. Then, by definition of Fourier transform and convolution in the distributional sense we have that

$$(u * \varphi)^\wedge(\psi) = (u * \varphi)(\hat{\psi}) = u(\tilde{\varphi} * \hat{\psi}) = u((\hat{\varphi}\psi)^\wedge) = (\hat{u}\hat{\varphi})(\psi)$$

This shows that $(u * \varphi)^\wedge = \hat{u}\hat{\varphi}$ in general.
Consider the function $\varphi_0(x) = e^{-\pi|x|^2}$ in $\mathcal{S}$. Since $u \in (L^2, L^2)$, Plancherel's theorem implies that $\Phi_0 = (u * \varphi_0)^\wedge = \hat{u}\hat{\varphi}_0 = \hat{u}\varphi_0$ belongs to $\mathcal{L}_2(\mathbb{R}^n)$. Let $b(x) = e^{\pi|x|^2}\Phi_0(x) = \Phi_0(x)/\varphi_0(x)$. Let $\varphi \in \mathcal{S}$ and $\psi \in \mathcal{D}$, then

$$\begin{aligned} (\hat{u}\hat{\varphi})(\psi) &= \hat{u}(\hat{\varphi}\psi) = \hat{u}(\hat{\varphi}_0\hat{\varphi}\psi/\varphi_0) = (\hat{u}\varphi)(\hat{\varphi}\psi/\varphi_0) \\ &= \int_{\mathbb{R}^n} \Phi_0(x)\hat{\varphi}(x)e^{\pi|x|^2}\psi(x)\,dx \\ &= \int_{\mathbb{R}^n} b(x)\hat{\varphi}(x)\psi(x)\,dx \\ &= (b\hat{\varphi})(\psi) \end{aligned}$$

Taking φ such that the support of its Fourier transform contains the support of ψ we get that $\hat{u}(\psi) = b(\psi)$ for all $\psi \in \mathcal{D}$. The density of $\mathcal{D}$ in $\mathcal{S}$ implies that $\hat{u} = b$.

Since $u \in (L^2, L^2)$ there exists a constant $A > 0$ such that

$$\|b\hat{\varphi}\|_2 = \|(u * \varphi)^\wedge\|_2 = \|u * \varphi\|_2 \leq A\|\varphi\|_2$$

for all $\varphi \in \mathcal{S}$. Since $\mathcal{S}$ is dense in $\mathcal{L}_2(\mathbb{R}^n)$, it is now easy to show that $b \in L^\infty(\mathbb{R}^n)$ with $\|b\|_\infty \leq A$.
The converse is easy: if $\hat{u} = b \in L^\infty(\mathbb{R}^n)$ then

$$\|u * \varphi\|_2 = \|\hat{u}\hat{\varphi}\|_2 = \|b\hat{\varphi}\|_2 \leq \|b\|_\infty \|\hat{\varphi}\|_2 = \|b\|_\infty \|\varphi\|_2$$

This finishes the proof. □

For $p = 1 = q$ there is a nice characterization as well.

Theorem 4.10. *The distribution u belongs to (L_1, L_1) if and only if it is a finite Borel measure. In this case, the total variation of u equals the norm of the operator $B : L^1 \cap \mathcal{S} \to L^1$ defined by $B\varphi = u * \varphi$ for $\varphi \in \mathcal{S}$*

Proof. Consider the space $\mathcal{C}_o(\mathbb{R}^n)$ of continuous functions that vanish at infinity with the sup norm. Its dual space, according to the Riesz–representation theorem, is the normed space of finite Borel measures $M = M(\mathbb{R}^n)$ on $\mathbb{R}^n$: mapping the measure μ to the linear functional that assigns φ the value $\int_{\mathbb{R}^n} \varphi(x)\, d\mu(x)$. Note that $\mathcal{L}_1(\mathbb{R}^n)$ is naturally embedded into M by the map $f \mapsto f\, dx$.
On the other hand, we can topologize M with the weak*–topology and by Alaoglu's theorem, the unit ball $B = \{\mu : \|\mu\| \leq 1\}$ in M is compact with respect this topology; moreover, since $\mathcal{C}_0$ is separable, the unit ball is in fact metrizable (with the weak*–topology), thus sequentially compact.

Consider the family of L^1 functions $u_\varepsilon = u * W(\cdot, \varepsilon)$, $\varepsilon > 0$, where W is the Gauss–Weierstrass kernel. By hypothesis, there exists $A > 0$ such that $\|u_\varepsilon\|_1 \leq A\|W(\cdot, \varepsilon)\|_1 = A$. Thus, by the previous observations, there exists $\mu \in M$ and a sequence $\{\varepsilon_k\}$ converging to 0 such that $u_{\varepsilon_k} \to \mu$ as $k \to \infty$ in the weak*–topology. That is

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} \varphi(x)\, du_{\varepsilon_k}(x) = \int_{\mathbb{R}^n} \varphi(x)\, d\mu(x)$$

for any $\varphi \in \mathcal{S}$. Let $\varphi_\varepsilon(x) = \int_{\mathbb{R}^n} \varphi(x-t) * W(t, \varepsilon)\, dt$. By dominated convergence we have that $(D^\alpha \varphi_\varepsilon)(x) = \int_{\mathbb{R}^n} (D^\alpha \varphi)(x-t) W(t, \varepsilon)\, dt = (D^\alpha \varphi)_\varepsilon$ for any n–tuple α of nonnegative numbers. By the theorem of approximations to the identity, (case $p = \infty$) we have that $D^\alpha \varphi_\varepsilon(x)$ converges uniformly in x to $D^\alpha \varphi(x)$ as $\varepsilon \to 0$. Thus $\varphi_\varepsilon \to \varphi$ in $\mathcal{S}$ as $\varepsilon \to 0$. Hence, by continuity of u, $u(\varphi_\varepsilon) \to u(\varphi)$. Since $W(\cdot, \varepsilon) = \tilde{W}(\cdot, \varepsilon)$ we have

$$u(\varphi_\varepsilon) = u(W(\cdot, \varepsilon) * \varphi) = (u * W(\cdot, \varepsilon))(\varphi) = \int_{\mathbb{R}^n} \varphi(x) u_\varepsilon(x)\, dx$$

Taking $\varepsilon = \varepsilon_k$ and letting $k \to 0$ we obtain

$$u(\varphi) = \int_{\mathbb{R}^n} \varphi(x)\, d\mu(x)$$

This shows that $u = \mu \in M(\mathbb{R}^n)$. The rest of the proof is easy. □

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