Effective invariants of transverse knots

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Conference on contact and symplectic topology
Université de Nantes
17 June 2011

Partly based on:

- joint work with Tobias Ekholm, John Etnyre, and Michael Sullivan
- preliminary joint work with Dylan Thurston

These slides available at `http://www.math.duke.edu/~ng/nantes.pdf`.

Outline

Transverse knots

M cooriented contact 3-manifold with contact structure $\xi = \ker \alpha$.
Standard example: $M = \mathbb{R}^3$, $\alpha_{\text{std}} = dz - y\,dx$.

Definition

A knot K in (M, ξ) is transverse if $\alpha > 0$ along K (in particular, $K \pitchfork \xi$). Two transverse knots are transversely isotopic if they are isotopic through transverse knots.

Transverse classification problem

Classify transverse knots of some particular topological type.

We'll restrict our attention to $(\mathbb{R}^3, \xi_{\text{std}} = \ker \alpha_{\text{std}})$.

Relation to Legendrian knots

- There is a one-to-one correspondence

$$\{\text{transverse knots}\} \longleftrightarrow \{\text{Legendrian knots}\}/$$
$$(+\text{ Legendrian stabilization/destab}).$$

- In $\mathbb{R}^3$, the classical invariant (self-linking number) of T and the classical invariants (Thurston–Bennequin number and rotation number) of L are related by

$$sl(T) = tb(L) - rot(L).$$

Braids and transverse knots

Theorem (Bennequin 1983)

Any braid (conjugacy class) can be closed in a natural way to produce a transverse knot in $(\mathbb{R}^3, \xi_{std})$, and every transverse knot is transversely isotopic to a closed braid.

Braids and transverse knots

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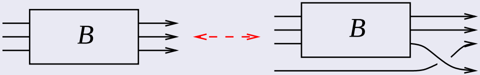
Transverse Markov Theorem (Orevkov–Shevchishin 01, Wrinkle 02)

Two braids represent the same transverse knot iff related by:

- *conjugation in the braid groups*
- *positive stabilization $B \longleftrightarrow B\sigma_n$:*



Cf. usual Markov Theorem: topological knots/links are equivalent to braids mod conjugation and positive/negative stabilization.

Transverse classification

If a transverse knot T is the closure of a braid B, the self-linking number of T is

$$sl(T) = w(B) - n(B)$$

where $w(B)$ = algebraic crossing number of B and $n(B)$ = braid index of B.

Definition

A topological knot is transversely simple if its transverse representatives are completely determined by self-linking number; otherwise transversely nonsimple.

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Definition

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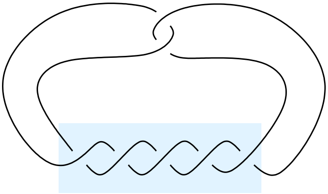
Examples of transversely simple knots:

- unknot (Eliashberg 1993)
- torus knots (Etnyre 1999) and the figure 8 knot (Etnyre–Honda 2000)
- some twist knots (Etnyre–N.–Vértesi 2010)

Transverse nonsimplicity

Examples of transversely nonsimple knots:

- $(2,3)$-cable of $(2,3)$ torus knot (Etnyre–Honda 2003) and other torus knot cables (Etnyre–LaFountain–Tosun 2011)
- some closed 3-braids (Birman–Menasco 2003, 2008)
- some twist knots (Etnyre–N.–Vértesi 2010): number of crossings in shaded region is odd and ≥ 5

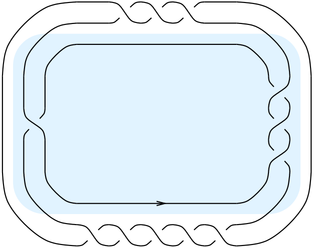


Transversely nonsimple knots: Birman–Menasco examples

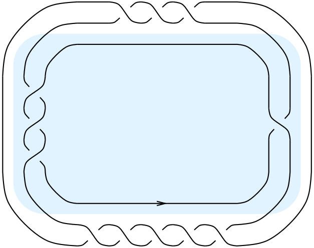
Birman–Menasco 2008: family of knots with braid index 3 that are transversely nonsimple. More precisely, they show that the transverse knots given by the closures of the 3-braids

$$\sigma_1^a \sigma_2^b \sigma_1^c \sigma_2^{-1}, \quad \sigma_1^a \sigma_2^{-1} \sigma_1^c \sigma_2^b,$$

which are related by a “negative flype”, are transversely nonisotopic for particular choices of (a, b, c).



$\sigma_1^5\sigma_2^3\sigma_1^3\sigma_2^{-1}$



$\sigma_1^5\sigma_2^{-1}\sigma_1^3\sigma_2^3$

Effective transverse invariants

Definition

A transverse invariant is effective if it can distinguish different transverse knots with the same self-linking number and topological type (i.e., prove that some topological knot is transversely nonsimple).

Not known to be effective:

- Plamenevskaya 2004: distinguished element in Khovanov homology
- Wu 2005: distinguished elements in Khovanov–Rozansky $\mathfrak{sl}_n$ homology
- N.–Rasmussen 2007: distinguished element in Khovanov–Rozansky HOMFLY-PT homology (known not to be effective)

Effective transverse invariants, continued

Known to be effective:

- Ozsváth–Szabó–Thurston 2006: HFK grid invariant: distinguished element in knot Floer homology via grid diagrams
- Lisca–Ozsváth–Stipsicz–Szabó 2008: LOSS invariant: distinguished element in knot Floer homology via open book decompositions
- Ekholm–Etnyre–N.–Sullivan 2010: transverse homology: filtered version of knot contact homology

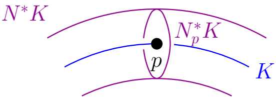
The conormal construction

Idea: use the cotangent bundle to turn smooth topology into symplectic/contact topology.

- M smooth manifold $\rightsquigarrow$ unit cotangent bundle ST^*M, which is naturally a contact manifold
- $K \subset M$ embedded submanifold $\rightsquigarrow$ conormal bundle

$$N^*K = \{(q,p) : q \in K, \langle p, v\rangle = 0 \,\forall\, v \in T_qK\} \subset ST^*M,$$

which is a Legendrian submanifold of ST^*M.



Smooth isotopy of $K \subset M$ results in *Legendrian* isotopy of $N^*K \subset ST^*M$.

Knot contact homology

$$(K \subset M \Longrightarrow N^*K \text{ Legendrian} \subset ST^*M \text{ contact})$$

Any *Legendrian-isotopy invariant* of N^*K is a *smooth-isotopy invariant* of K: for instance, Legendrian contact homology (Eliashberg–Hofer), where defined. For $M = \mathbb{R}^n$, $ST^*M = J^1(S^{n-1})$ and LCH is well-defined (Ekholm–Etnyre–Sullivan 05).

Definition

$K \subset \mathbb{R}^n$. The knot contact homology of K is the Legendrian contact homology of $N^*K \subset ST^*\mathbb{R}^n$,

$$HC_*(K) := LCH_*(N^*K).$$

In particular, for a knot $K \subset \mathbb{R}^3$, $HC_*(K)$ is a smooth knot invariant.

Form for knot contact homology

$$(K \subset M \Longrightarrow N^*K \text{ Legendrian} \subset ST^*M \text{ contact})$$

For a knot $K \subset \mathbb{R}^3$, the LCH complex for N^*K is a differential graded algebra

$$(CC_*(K), \partial)$$

generated by Reeb chords for N^*K, over the group ring

$$R := \mathbb{Z}[H_1(N^*K)] \cong \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}].$$

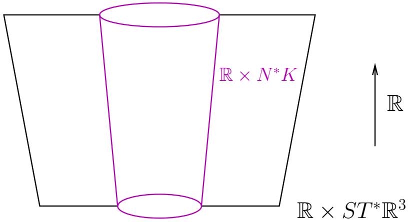
The differential counts holomorphic disks in $ST^*\mathbb{R}^3$ with boundary on N^*K.

Theorem (N. 2003, 2004, Ekholm–Etnyre–N.–Sullivan in preparation)

There is a purely algebraic/combinatorial expression for the DGA $(CC_*(K), \partial)$.

Holomorphic disks counted in knot contact homology

The symplectization $\mathbb{R} \times ST^*\mathbb{R}^3$, and the Lagrangian cylinder $\mathbb{R} \times N^*K$ in the symplectization:



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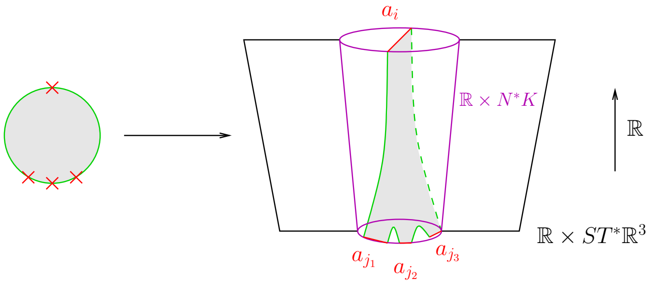


This holomorphic disk contributes

$$\partial(a_i) = a_{j_1} a_{j_2} a_{j_3} + \cdots$$

where $a_i, a_{j_1}, a_{j_2}, a_{j_3}$ are Reeb chords of N^*K.

Properties of knot contact homology

(Recall: knot contact homology $HC_*(K)$ = Legendrian contact homology of conormal $N^*K \subset ST^*\mathbb{R}^3$.)

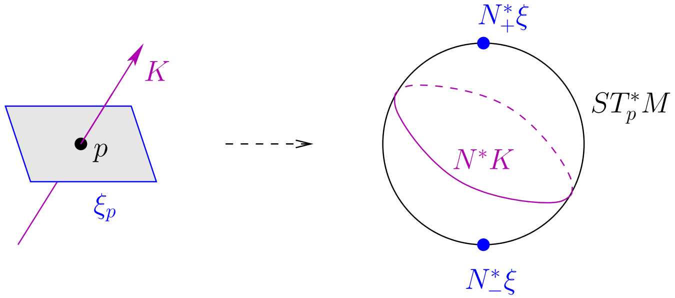
Theorem (N. 2004)

- *$HC_*(K)$ can be combinatorially shown to be a knot invariant, supported in degrees $* \geq 0$.*
- *(Linearized) $HC_1(K)$ encodes the Alexander polynomial $\Delta_K(t)$.*
- *$HC_0(K)$ detects the unknot.*

Lifting a contact structure

Given a contact manifold (M, ξ), the contact structure ξ itself has a conormal lift to ST^*M:

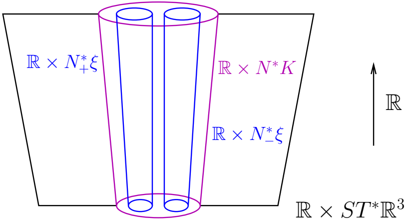
$$N^*\xi = N^*_+\xi \cup N^*_-\xi = \{(q,p) \in ST^*M : \langle p, v\rangle = 0 \,\forall\, v \in \xi_q\}.$$



If K is transverse to ξ, then the conormal lifts of K and ξ are disjoint: $N^*K \cap N^*_\pm\xi = \emptyset$.

Filtering the LCH differential

If K is transverse in $(\mathbb{R}^3, \xi)$, we can filter the LCH differential for N^*K by counting intersections with the holomorphic 4-manifolds $\mathbb{R} \times N^*_\pm \xi$ in the symplectization $\mathbb{R} \times ST^*\mathbb{R}^3$.



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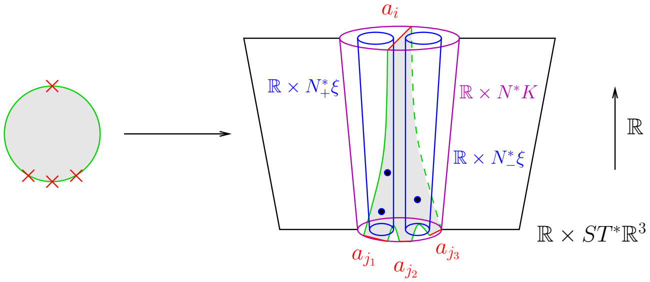


This lifts the LCH complex from a DGA $(\mathcal{A}, \partial)$ over $R = \mathbb{Z}[\lambda^{\pm1}, \mu^{\pm1}]$ to a DGA $(\mathcal{A}, \partial^-)$ over $R[U, V]$: e.g.

$$\partial^-(a_i) = U^2 V^1 a_{j_1} a_{j_2} a_{j_3} + \cdots .$$

Transverse homology

Definition

The (minus) transverse complex of a transverse knot $K \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ is the LCH algebra $(CT_*^-(K) = \mathcal{A}, \partial^-)$ over the base ring $R[U, V] = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}, U, V]$, with the differential ∂^- filtered by intersections with $N_\pm^* \xi$.

This can be viewed as a bi-filtered version of knot contact homology.

Theorem (Ekholm–Etnyre–N.–Sullivan 2010)

- *There is a combinatorial formula for $(CT_*^-(K), \partial^-)$ in terms of a braid representative of K.*
- *The transverse homology of K, $HT_*^-(K) = H_*(CT^-(K), \partial^-)$, is a transverse invariant.*

Flavors of transverse homology

From $(CT^-(K), \partial^-)$ chain complex over $R[U, V]$ (with $R = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$), obtain:

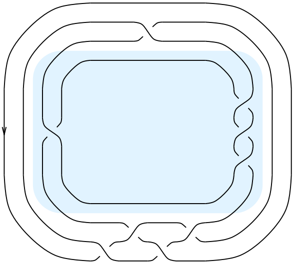
- $(\widehat{CT}_*(K), \hat{\partial})$ chain complex over R, by setting $(U, V) = (0, 1)$ or $(1, 0)$
- $(CT_*^\infty(K), \partial^\infty)$ chain complex over $R[U^{\pm 1}, V^{\pm 1}]$, by tensoring with $R[U^{\pm 1}, V^{\pm 1}]$
- $(CC_*(K), \partial)$ chain complex over R, by setting $(U, V) = (1, 1)$

$\widehat{HT}_*(K)$ is a transverse invariant, while $HT_*^\infty(K)$ and $HC_*(K)$ are topological invariants (the latter is knot contact homology).

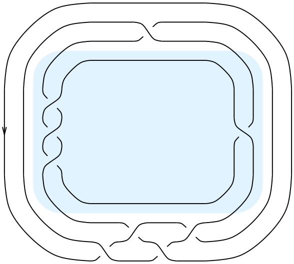
Theorem (N. 2010)

$\widehat{HT}_0(K)$ is an effective transverse invariant.

Example: $m(7_6)$ knot



$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}(\sigma_3^3\sigma_2\sigma_3^{-1})$



$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}(\sigma_3^{-1}\sigma_2\sigma_3^3)$

These two transverse $m(7_6)$ knots can be distinguished by $\widehat{HT}_0$: count number of augmentations (ring homomorphisms)

$$\widehat{HT}_0 \to \mathbb{Z}/3.$$

This is an effective technique for distinguishing other transverse knots, as long as braid index $\lesssim 4$.

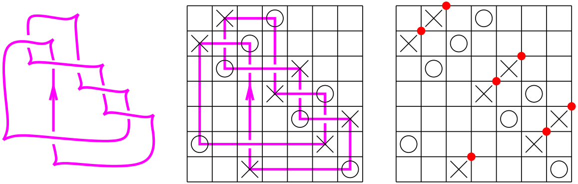
HFK grid invariant

Ozsváth–Szabó–Thurston 2006:

transverse knot T of topological type K

$$\downarrow$$

distinguished element $\theta^-(T) \in HFK^-(m(K))$.



In combinatorial model for CFK via grid diagrams (Manolescu–Ozsváth–Sarkar), $\theta^-(T)$ is the generator given by the upper-right corners of the X's for a Legendrian approximation of T.

HFK grid invariant, continued

Result (after mapping $HFK^- \to \widehat{HFK}$) for T transverse of type K:

$$\widehat{\theta}(T) \in \widehat{HFK}_{sl(T)+1}(m(K), \tfrac{sl(T)+1}{2}).$$

Theorem (Ozsváth–Szabó–Thurston 2006)

The HFK grid invariant $\widehat{\theta}$ is a transverse invariant.

Crude way to apply $\widehat{\theta}$: if T_1, T_2 are transverse knots with $\widehat{\theta}(T_1) = 0$ and $\widehat{\theta}(T_2) \neq 0$, then they're distinct.

Theorem (N.–Ozsváth–Thurston 2007)

The HFK grid invariant $\widehat{\theta}$ is an effective transverse invariant.

E.g., can be used to recover Etynre–Honda's result that the $(2,3)$-cable of the $(2,3)$ torus knot is transversely nonsimple.

Limitations of crude approach

$$\widehat{\theta}(T) \in \widehat{HFK}_{sl(T)+1}(m(K), \tfrac{sl(T)+1}{2}) :$$

- If this group is 0, then $\widehat{\theta}(T) = 0$ carries no information.
- If $\widehat{\theta}(T_1), \widehat{\theta}(T_2) \neq 0$, how to tell them apart?

Slightly more precise statement of invariance:

Theorem (Ozsváth–Szabó–Thurston 2006)

If T_1, T_2 are isotopic transverse knots and G_1, G_2 are grid diagrams of corresponding Legendrian approximations, then the transverse isotopy gives a sequence of grid moves from G_1 to G_2 inducing a combinatorially-defined isomorphism

$$\phi : \widehat{HFK}(G_1) \to \widehat{HFK}(G_2)$$

and $\phi(\widehat{\theta}(G_1)) = \widehat{\theta}(G_2)$.

Enter naturality

Theorem (Thurston et al., in progress)

(roughly speaking) Let G_1, G_2 be grid diagrams for the same topological knot, and let γ be a sequence of grid moves from G_1 to G_2. Then the isomorphism

$$\gamma_* : HFK^-(G_1) \to HFK^-(G_2)$$

depends only on the homotopy class of the path $\gamma \subset \{smooth\ knots\}$.

Definition

Let K be an oriented topological knot. The mapping class group of K is

$$MCG(K) = \pi_1(\{\text{smooth knots isotopic to } K\}).$$

Can use naturality in conjunction with $\widehat{\theta}$.

Naturality and the HFK grid invariant

Corollary

Let T_1, T_2 be transverse of type K with $MCG(K) = 1$, and let G_1, G_2 be grid diagrams for T_1, T_2. If T_1, T_2 are transversely isotopic, then for any sequence γ of grid diagrams from G_1 to G_2,

$$\gamma_*(\widehat{\theta}(G_1)) = \widehat{\theta}(G_2).$$

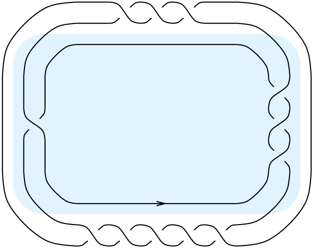
Theorem (N.–Thurston 2011, preliminary)

The Birman–Menasco pair

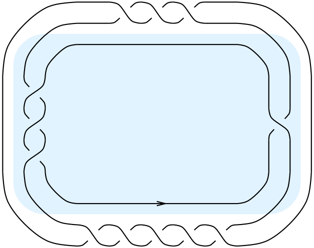
$$\sigma_1^5\sigma_2^3\sigma_1^3\sigma_2^{-1} \quad \textit{and} \quad \sigma_1^5\sigma_2^{-1}\sigma_1^3\sigma_2^3$$

can be distinguished by $\widehat{\theta}$.

Birman–Menasco transverse knots



$\sigma_1^5\sigma_2^3\sigma_1^3\sigma_2^{-1}$



$\sigma_1^5\sigma_2^{-1}\sigma_1^3\sigma_2^3$

These are of topological type $11a_{240}$, and $MCG(11a_{240}) = 1$. The $\widehat{\theta}$ invariants constitute distinct nonzero elements of

$$\widehat{HFK}_8(11a_{240}, 4) \cong (\mathbb{Z}/2)^2.$$

This argument can be extended to other Birman–Menasco pairs (possibly $\sigma_1^a\sigma_2^b\sigma_1^c\sigma_2^{-1}, \sigma_1^a\sigma_2^{-1}\sigma_1^c\sigma_2^b$ for $a, b, c \geq 3$ with $a \neq c$), but not all of them.

Transverse mapping class group

Definition

Let K be a transverse knot. The transverse mapping class group of K is

$$TMCG(K) = \pi_1(\{\text{transverse knots transversely isotopic to } K\}).$$

For a transverse knot K, there is an obvious map

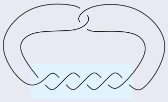
$$TMCG(K) \to MCG(K).$$

Naturality and $\widehat{\theta}$ can be used to show that this map is not an isomorphism for some transverse knots K.

Transverse mapping class group, continued

Theorem (N.–Thurston 2011, preliminary)

Consider any twist knot where the number of crossings in the shaded region is odd and ≥ 3.



There is a transverse knot K *of this topological type such that the map*

$$TMCG(K) \to MCG(K)\ (\cong \mathbb{Z}/2)$$

is not surjective.

Cf. Kálmán 2004: there are Legendrian knots K for which the map $LMCG(K) \to MCG(K)$ is not *injective*.

Comparison of transverse invariants

Legendrian knot atlas (Chongchitmate–N. 2010): 13 knots of arc index ≤ 9 are conjectured to be transversely nonsimple.

Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK					
HT					
Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK					
HT					
Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK					
HT					

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HFK		✓		✓	
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HFK					
HT					

N.–Ozsváth–Thurston 2007, using HFK grid invariant

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Chongchitmate–N. 2010, using HFK grid invariant

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HFK		✓		✓	
HT					
Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK	✓		✓	✓	
HT					

Ozsváth–Stipsicz 2008, using LOSS invariant

Comparison of transverse invariants

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HFK	$\checkmark$			$\checkmark$	
HT					

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HT					

N.–Thurston 2011, using HFK grid invariant and naturality

Comparison of transverse invariants

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Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK	✓	×	×	✓	×
HT					

Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK	✓	✓	×	✓	
HT					

Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK	✓	×	✓	✓	
HT					

HFK invariants don't work: $\widehat{HFK} = 0$ in relevant bidegree.

Comparison of transverse invariants

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Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK	✓	×	×	✓	×
HT	✓	✓	✓		✓

Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK	✓	✓	×	✓	
HT		✓	✓	✓	

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HFK	✓	×	✓	✓	
HT	✓		✓	✓	

N. 2010, using transverse homology

Comparison of transverse invariants

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HT	✓	✓	✓	?	✓
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HFK	✓	✓	×	✓	
HT	?	✓	✓	✓	
Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK	✓	×	✓	✓	
HT	✓	?	✓	✓	

These are "transverse mirrors", as are the Birman–Menasco knots.

Comparison of transverse invariants

Legendrian knot atlas (Chongchitmate–N. 2010): 13 knots of arc index ≤ 9 are conjectured to be transversely nonsimple.

Knot	$m(7_2)$	$m(7_6)$	9_{44}	$m(9_{45})$	9_{48}
HFK	✓	×	×	✓	×
HT	✓	✓	✓	?	✓
Knot	10_{128}	$m(10_{132})$	10_{136}	$m(10_{140})$	
HFK	✓	✓	×	✓	
HT	?	✓	✓	✓	
Knot	$m(10_{145})$	10_{160}	$m(10_{161})$	$12n_{591}$	
HFK	✓	×	✓	✓	
HT	✓	?	✓	✓	