

Math 621 — part 2 — 2018 edition

Note Title

2/14/2017

(red date as 2018; magenta as 2017)

Riemannian Metrics

Def A Riemannian metric is a $(0,2)$ -tensor g , i.e. a smoothly varying

$$g: T_x M \otimes T_x M \rightarrow \mathbb{R}$$
$$v \otimes w \mapsto g(v, w) \text{ or } \langle v, w \rangle$$

that is.

- SYMMETRIC : $g(v, w) = g(w, v)$

- POSITIVE DEFINITE : $g(v, v) \geq 0$, equality $\Leftrightarrow v = 0$.

In coordinates, write $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$.

Def A Riemannian manifold is (M, g) , M —smooth mfd,

g = Riem. metric.

When are two such mfd's the "same"?

Def An isometry between (M, g) and (N, h) is a diffeomorphism

$\varphi: M \rightarrow N$ such that $g = \varphi^* h$, i.e. :

$$h(\varphi_* v, \varphi_* w) = g(v, w) \quad \forall v, w \in T_p M$$

$\varphi: M \rightarrow N$ is a local isometry at $p \in M$ if \exists nbhd U of p with
 $\varphi|_U: U \rightarrow \varphi(U)$ isometry.

Ex 1. \mathbb{R}^n , $g = \text{standard inner product}$: $g_{ij} = \delta_{ij}$ Kronecker delta.
Euclidean space.

Ex 2. Lie groups.

Def A Riem metric \langle , \rangle on G is left invariant if L_h is an isometry $\forall h \in G$.

$$\forall h \in G, \forall v, w \in T_g G, \quad \langle v, w \rangle = \underset{\text{at } g}{\langle (L_h)_* v, (L_h)_* w \rangle} \underset{\text{at } hg}{\langle}$$

Similarly: right invariant. If both, then biinvariant.

Easy to construct left invt metric: given an inner product \langle , \rangle on $g = T_e G$, define $\langle v, w \rangle = \underset{\text{at } g}{\langle (L_g)_* v, (L_g)_* w \rangle}$.

Fact: Any compact lie group has a biinvariant metric
(See doCarmo p. 46 #7). (not true in general)

Prop Let \langle , \rangle be the left invt metric on G induced by \langle , \rangle on g .

Then \langle , \rangle is biinvariant \iff

$$0 = \langle [X, Y], Z \rangle_e + \langle Y, [X, Z] \rangle_e \quad \forall X, Y, Z \in g.$$

Pf. \Rightarrow . Let $X, Y, Z \in g$. For $t \in \mathbb{R}$, recall $\exp(tX) = \varphi_t(e)$; and $[X, Y] = \frac{d}{dt}|_{t=0} \text{Ad}(\exp(tX)) Y$ where $\text{Ad}(h) = (R_{h^{-1}})_*$.

$$\begin{aligned} \text{Then } \langle Y, Z \rangle &= \langle (R_{h^{-1}})_* Y, (R_{h^{-1}})_* Z \rangle & h = \exp(tX) \\ &= \langle \text{Ad}(\exp(tX)) Y, \text{Ad}(\exp(tX)) Z \rangle \end{aligned}$$

$$\text{and } \frac{d}{dt}: \quad 0 = \langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle. \\ \iff : \text{HW.}$$

Ex 3. Given an immersion $f: M \rightarrow N$ (often use $N = \mathbb{R}^{n+k}$),

a Riem. metric $\langle \cdot, \cdot \rangle_N$ on N induces one $\langle \cdot, \cdot \rangle_M$ on M :

$$\langle v, w \rangle_M = \langle f_* v, f_* w \rangle_N.$$

Note immersion $\Rightarrow df = f_*: T_p M \rightarrow T_{f(p)} N$ is injective.

Thus $\langle \cdot, \cdot \rangle_M$ is pos def since if $\langle v, v \rangle_M = 0$ then $f_* v = 0 \Rightarrow v = 0$.

So e.g. embedded submfds $M \hookrightarrow \mathbb{R}^{n+k}$ have metric induced from Euclidean metric on \mathbb{R}^{n+k} .

ex: $S^n \subset \mathbb{R}^{n+1}$ unit sphere inherits the "round metric".

$$\{x_1^2 + \dots + x_{n+1}^2 = 1\}$$

(for S^1 , this agrees with flat metric on $\mathbb{R}/2\pi$)

More general way to get submfds:

$$h: N^{n+k} \rightarrow P^k \text{ smooth.}$$

- $p \in N$ is a critical point if d_h not surjective
- $q \in P$ is a critical value if $q = h(p)$, some critical p
- $q \in P$ is a regular value if not critical value.

$\cancel{\text{P}}$ Prop $q = \text{regular value} \Rightarrow h^{-1}(q) \subset N$ is a smooth n -manifold.

So a metric on N induces one on $M \hookrightarrow N$.

If $p \in M \Rightarrow d_{h,p}: T_p N \rightarrow T_{h(p)} P$ is surjective: can choose coords $x_1, \dots, x_n, y_1, \dots, y_k$ near p st. $h = (h_1, \dots, h_k)$, $\begin{pmatrix} \frac{\partial(h_1, \dots, h_k)}{\partial(y_1, \dots, y_k)} \end{pmatrix}$ is non-singular.

Implicit Function Thm $\Rightarrow \exists g: \mathbb{R}^n \rightarrow \mathbb{R}^k$ with $h(x, g(x)) = q$,
and this gives a chart

$$\begin{aligned} \mathbb{R}^n &\rightarrow M \\ x &\mapsto (x, g(x)) \end{aligned} \quad \square$$

$$\rightarrow \left\{ \boxed{\frac{\partial h}{\partial y}} \right\}_{n+k}$$

Ex 4 Product manifolds.

$(M_1, g_1), (M_2, g_2)$ Riem mfd $\rightarrow M_1 \times M_2$.

recall $T_{(p_1, p_2)}(M_1 \times M_2) \cong T_{p_1}M_1 \oplus T_{p_2}M_2$.

If $(v_1, v_2), (w_1, w_2) \in T_{(p_1, p_2)}(M_1 \times M_2)$ then define

$$g((v_1, v_2), (w_1, w_2)) = g_1(v_1, w_1) + g_2(v_2, w_2).$$

easy to check: this is a metric.

So eg. $T^n = \underbrace{S^1 \times \dots \times S^1}$ has a metric induced from S^1

(the book calls it the "flat metric").

Prop Any smooth mfd has a Riem metric.

Def $\{V_\alpha \subset M\}$ is locally finite if $\forall p \in M$, $p \in V_\alpha$ for only finitely many α .

Def $f \in C^\infty(M)$. The support of f is $\text{supp } f = \overline{\{p \in M \mid f(p) \neq 0\}}$.

Def $\{V_\alpha\}$ locally finite open cover of M . A partition of unity

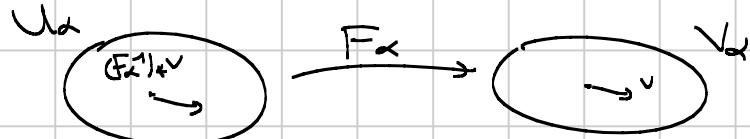
subordinate to $\{V_\alpha\}$ is a collection $f_\alpha \in C^\infty(M)$ s.t.

1. $f_\alpha(p) \geq 0 \forall p$
2. $\text{supp } f_\alpha \subset V_\alpha$
3. $\sum_\alpha f_\alpha(p) = 1 \quad \forall p$ (note finite sum for each p).

Then M smooth mfd, $\{V_\alpha\}$ open cover. Then \exists locally finite open cover $\{V'_\beta\}$ subordinate to $\{V_\alpha\}$, i.e. $\forall \beta \quad V'_\beta \subset V_\alpha$ for some α , and a partition of unity subordinate to $\{V'_\beta\}$.

Pf of Prop let $\{(F_\alpha, U_\alpha, V_\alpha)\}$ be an atlas for M . By them, assume $\{V_\alpha\}$ locally finite and $f_\alpha = \text{partition of unity subordinate to } \{V_\alpha\}$.

Each α determines a metric \langle , \rangle_α on V_α : $\langle v, w \rangle_\alpha = \langle F_\alpha^* v, F_\alpha^* w \rangle$



Then define \langle , \rangle on M by

$$v, w \in T_p M \Rightarrow \langle v, w \rangle = \sum_{\alpha} f_{\alpha}(p) \langle v, w \rangle_{\alpha}. \quad (\text{note: finite sum})$$

Easy to check: symmetric, bilinear, pos def. D

Uses for metrics -

- Length of curve. $\gamma: [a, b] \rightarrow M$ piecewise smooth.
→ length $l(\gamma) = \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} dt \quad (\gamma'(t) \in T_{\gamma(t)} M)$
Usual pf: indep of parametrization.
- Ism between tangent + cotangent.
 $p \in M \rightarrow g(p): T_p M \otimes T_p M \rightarrow \mathbb{R}$
→ $\tilde{g}(p): T_p M \rightarrow T_p^* M$.
 g Pos def \Rightarrow isomorphism. So set $\tilde{g}: TM \xrightarrow{\sim} T^* M$ bundle ism.
Note: works for any nondeg $(0, 2)$ -tensor.
- Volume form. Assume M oriented.
 $g \mapsto \omega \in \Omega^n(M)$ given in local coords by
 $\omega = \sqrt{\det(g_{ij})} dx_1 \wedge \dots \wedge dx_n. \quad (g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}))$.

Check well-defined: change of coords $x_1 \dots x_n \rightsquigarrow y_1 \dots y_n$.

$$\frac{\partial}{\partial y_i} = \sum \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial x_j}.$$

Write Jacobian $\left(\frac{\partial x_i}{\partial y_j} \right) = M$: then if $\tilde{g}_{ij} = g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right)$, the

$$(\tilde{g}_{ij}) = M(g_{ij}) M^T.$$

$$\tilde{g}_{ij} = \sum \frac{\partial x_k}{\partial y_i} \frac{\partial x_k}{\partial y_j} g_{kk}.$$

Also $\begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix} = M^T \begin{pmatrix} dy_1 \\ \vdots \\ dy_n \end{pmatrix} \Rightarrow dx_1 \wedge \dots \wedge dx_n = (\det M) dy_1 \wedge \dots \wedge dy_n$.

$$\Rightarrow \sqrt{\det \tilde{g}_{ij}} dy_1 \wedge \dots \wedge dy_n = (\det M) \sqrt{\det g_{ij}} dy_1 \wedge \dots \wedge dy_n$$

(if $\det M > 0$)

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Affine Connections

Motivation:

① How to take directional derivatives of vector fields?

Would like: X tangent vector, $Y = \sum b_i \frac{\partial}{\partial x_i}$

\Rightarrow "directional derivative" " $X(Y) = \sum X(b_i) \frac{\partial}{\partial x_i}$ ".

Problem: not coord independent.

$$\text{ex. } M = \mathbb{R} \quad X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial x} \Rightarrow "X(Y)" = 0.$$

But: another coord system $y = x^3$.

$$\frac{\partial}{\partial x} = \frac{dy}{dx} \frac{\partial}{\partial y} = 3y^{2/3} \frac{\partial}{\partial y}$$

$$\Rightarrow "X(Y)" = 3y^{2/3} \frac{\partial}{\partial y} (3y^{4/3}) \frac{\partial}{\partial y} = 6y^{10/9} \frac{\partial}{\partial y} \neq 0.$$

② What's the analogue of a straight line in a mfld M ?

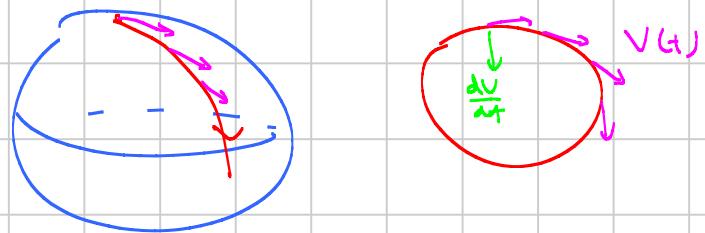
Need notion of "parallel" vector along a curve.



γ path in M , V = vectn field along γ i.e. $V(t) \in T_{\gamma(t)} M$.

What does it mean for V to be parallel? " $\frac{dV}{dt} = 0$ "

For $M \subset \mathbb{R}^3$ surface, $V(t) \in \mathbb{R}^3$ so $\frac{dV}{dt}$ is a vectn in \mathbb{R}^3 but not nec. in $T_{\gamma(t)} M$.



May & define

$\frac{dV}{dt}$ = orthogonal projection of $V'(t)$ to $T_{\gamma(t)} M$.

In general:

Def An affine connection ∇ on M is a map

$$\begin{aligned}\nabla: \text{Vect } M \times \text{Vect } M &\rightarrow \text{Vect } M \\ (X, Y) &\mapsto \nabla_X Y\end{aligned}$$

Satisfying:

1. $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$; $\nabla_X (Y_1+Y_2) = \nabla_X Y_1 + \nabla_X Y_2$
2. $\nabla_{fX} Y = f \nabla_X Y$
3. $\nabla_X (fY) = f \nabla_X Y + X(f)Y$.

Note: for fixed Y , the map $\nabla_Y: \text{Vect } M \rightarrow \text{Vect } M$ is a tensor

Interlude: Cords and indices

Convention: write x^1, \dots, x^n for cords.

$T_p M$ send by $\frac{\partial}{\partial x^i} = \partial_i, \dots, \frac{\partial}{\partial x^n} = \partial_n$; $T_p^* M$ send by dx^1, \dots, dx^n .

Vector field $\sum_i a^i \partial_i = a^i \partial_i$; 1-form $\sum_i b_i dx^i = b_i dx^i$.

Einstein summation notation: sum over repeated indices (one upper, one lower)

$$X = a^i \partial_i \Rightarrow Xf = \sum_i a^i \frac{\partial f}{\partial x^i} = a^i \partial_i f.$$

Metric $g_{ij} = \langle \partial_i, \partial_j \rangle$; $g = g_{ij} dx^i \otimes dx^j$.

for future use: write inverse matrix to (g_{ij}) as (g^{ij}) ; write $g_{ik} g^{kj} = \delta^j_i$.

Write $\nabla_i = \nabla_{\partial/\partial x^i}$.

Connection in coordinates

Def x^1, \dots, x^n coords. The Christoffel symbol Γ_{ij}^k , $1 \leq i, j, k \leq n$, is defined by

$$\nabla_{\partial/\partial x^i} \left(\frac{\partial}{\partial x^j} \right) = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k} \text{ i.e. } \nabla_i \partial_j = \Gamma_{ij}^k \partial_k.$$

$C^\infty(U)$

Connections are local operators (excl) and on a chart,

Γ_{ij}^k determine the connection.

if ∇ = connection then

$$\nabla_{a^i \partial_i} (b^j \partial_j) = a^i \nabla_i (b^j \partial_j) = a^i (b^j \nabla_i \partial_j + (\partial_i b^j) \partial_j) = a^i b^j \Gamma_{ij}^k \partial_k + a^i (\partial_i b^j) \partial_j.$$

Conversely can define $\nabla_x Y$ & X, Y by the formula \rightarrow

and check that this gives a connection.

$\text{Ex } M = \mathbb{R}^n, \Gamma_{ij}^k = 0 \quad \forall i, j, k.$

$$X = a^i \partial_i, Y = b^j \partial_j \Rightarrow D_X Y = a^i (\delta_i^j) \partial_j = X(b^j) \partial_j$$

(note in a different coord system, Γ_{ij}^k might not be 0).

3/2 □

Now: given $\gamma = \text{curve in } M, V = \text{vector field along } \gamma$, use D

to define $\frac{dV}{dt}$. ("covariant der. along a curve")



Prop M smooth, D = affine connection.

Then $\exists!$ way to associate to a curve $\gamma(t)$ and a vector field V along γ , another vector field $\frac{dV}{dt}$ along γ , s.t..

$$\textcircled{1} \quad \frac{D}{dt} (V + w) = \frac{DV}{dt} + \frac{Dw}{dt}$$

$$\textcircled{2} \quad \frac{D}{dt} (fV) = f \frac{DV}{dt} + \frac{df}{dt} V$$

\textcircled{3} If V extends to a vector field Y on M (or a neighborhood of γ) then

$$\frac{dV}{dt} = D_{\dot{\gamma}(t)} Y.$$

Pf Write $\gamma(t) = (x^1(t), \dots, x^n(t)) \Rightarrow \gamma'(t) = (x^i)' \partial_i$.

Uniqueness: by \textcircled{3},

$$\frac{D}{dt} (\partial_j) = D_{(x^i)'} \partial_i \partial_j = (x^i)' \Gamma_{ij}^k \partial_k$$

Then for general vector field along γ , $V = V^j(t) \partial_j$,

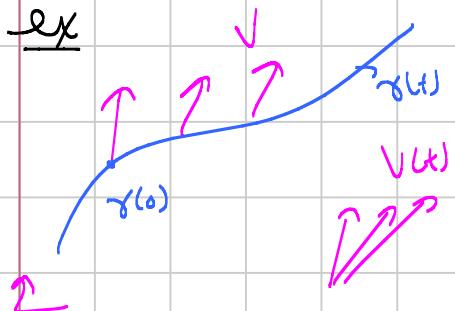
$$\frac{dV}{dt} = V^j(t) \frac{D}{dt} \partial_j + \frac{dV^j}{dt} \partial_j \quad (\textcircled{1} \text{ and } \textcircled{2})$$

$$= (x^i)' V^j(t) \Gamma_{ij}^k \partial_k + \frac{dV^j}{dt} \partial_j$$

$$\textcircled{3} \quad \boxed{\frac{dV}{dt} = \left(\frac{dV^k}{dt} + \frac{dx^i}{dt} V^j \Gamma_{ij}^k \right) \partial_k}.$$

Existence: define $\frac{dV}{dt}$ by \textcircled{3}; check \textcircled{2} (\textcircled{1}, \textcircled{3} obvious).

Note: on overlapping charts, values must agree by uniqueness. D



$$\frac{DV}{dt} \Big|_{t=0} = D_{\gamma'(0)} V = a^i (\partial_i \cdot b^j) \partial_j = a^i \frac{\partial b^j}{\partial x^i} \frac{\partial}{\partial x^j}$$

$$V = b^j \partial_j$$

$$\gamma'(0) = a^i \partial_i$$

$$V(\gamma(t)) = b^j(\gamma(t)) \partial_j$$

Parallel transport: $\gamma(t)$ curve, V = vector field along γ .

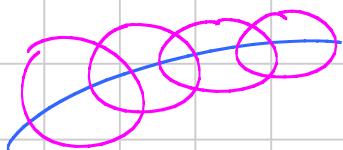
Def V is parallel if $\frac{DV}{dt} \Rightarrow 0$.

Prop $V_0 \in T_{\gamma(0)} M$. $\exists!$ parallel vector field $V(t)$ wth $V(0)=V_0$:
this is called "parallel transport" of V_0 along γ .

Pf Suffices to prove in a coord chart: then cover γ by overlapping charts.

We want to find

$$V(t) = V^i(t) \partial_i \quad \text{satisfying:}$$



$$\left(\underbrace{\frac{dV^k}{dt} + \frac{dx^i}{dt} V^j \Gamma_{ij}^k}_{=0 \forall k} \right) \partial_k = 0$$

$$\Leftrightarrow \frac{dV^k}{dt} = -V^j(t) \frac{dx^i}{dt} \Gamma_{ij}^k.$$

This is a system of n 1st order diff'ls in n variables $\Rightarrow \exists!$ solution given initial conditions. \square

Important example: $M \subset \mathbb{R}^{n+k}$.

Define affine connection $\bar{\nabla}$ on \mathbb{R}^{n+k} as before.
This induces an affine connection ∇ on M :

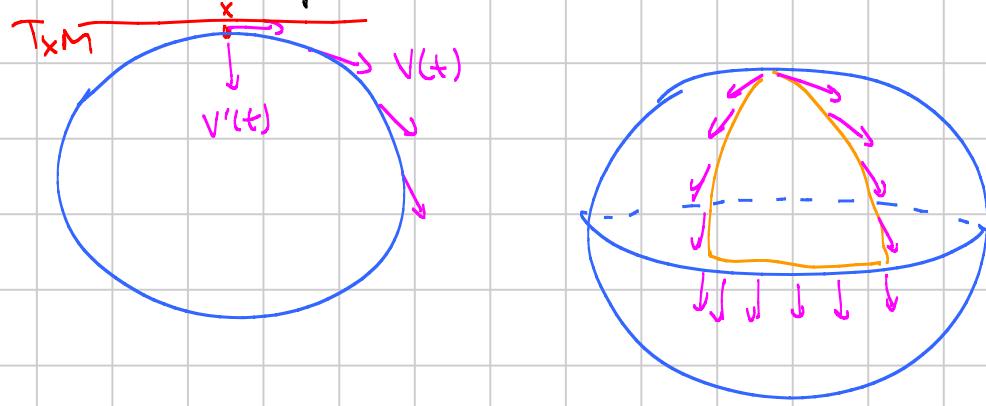
$X, Y \in \text{Vect}(M)$, expanded to $\bar{X}, \bar{Y} \in \text{Vect}(\mathbb{R}^{n+k}) \rightarrow \bar{\nabla}_X Y = (\bar{\nabla}_{\bar{X}} \bar{Y})^T$: orthogonal projection



$$T_x \mathbb{R}^{n+k} \rightarrow T_x M.$$

Check: this is an affine connection.

Parallel transport: $\frac{DV}{dt} = 0 \rightarrow$ if $V(t) \in \mathbb{R}^{n+k}$ then $(V'(t))^T = 0$.



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Levi-Civita Connection

g = Riemann metric on M

Def. A connection ∇ on M is compatible with g if $\forall X, Y, Z \in \text{Vect}(M)$,

$$X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Prop γ -curve on M , D compatible with $g = \langle \cdot, \cdot \rangle$.

1. If V, W are vector fields along γ , then

$$\frac{d}{dt} \langle V, W \rangle = \left\langle \frac{DV}{dt}, W \right\rangle + \left\langle V, \frac{DW}{dt} \right\rangle.$$

2. If V = parallel vector field along γ , then $|V(t)| = \langle V(t), V(t) \rangle^{1/2}$ is constant.

Pf 1. Extend V/W to vector fields near γ , and use $\frac{DV}{dt} = D_{\dot{\gamma}(t)} V$ etc.
2. Clear. \square

Def A connection D on M is torsion-free ("symmetric") if

$$D_X Y - D_Y X = [X, Y] \quad \forall X, Y \in \text{Vect } M.$$

Note: in coords $(\Gamma_{ij}^k - \Gamma_{ji}^k) \partial_k = \nabla_i \partial_j - \nabla_j \partial_i = [\partial_i, \partial_j] = 0 \Rightarrow \Gamma_{ij}^k = \Gamma_{ji}^k$.

Thm M Riem. There exists a unique connection that is torsion-free and compatible with g , "Levi-Civita connection".

Pf Uniqueness: $\oplus \quad X \langle Y, Z \rangle = \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle$

$$\oplus \quad Y \langle Z, X \rangle = \langle D_Y Z, X \rangle + \langle Z, D_Y X \rangle$$

$$\ominus \quad Z \langle X, Y \rangle = \langle D_Z X, Y \rangle + \langle X, D_Z Y \rangle$$

$$(X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle) = 2 \langle D_X Y, Z \rangle + \langle [X, Z], Y \rangle + \langle X, [Y, Z] \rangle - \langle Z, [X, Y] \rangle$$

so $\langle D_X Y, Z \rangle$ is determined $\forall X, Y, Z \Rightarrow D_X Y$ is determined.

Existence: HW. \square

In coord: choose $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$.

$$\langle D_X Y, Z \rangle = \Gamma_{ij}^m \langle \partial_m, \partial_k \rangle = \Gamma_{ij}^m g_{mk}$$

$$\Rightarrow \Gamma_{ij}^m g_{mk} = \frac{1}{2} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$$

$$\Rightarrow \Gamma_{ij}^k = \Gamma_{ij}^m g_{mk} g^{lk}$$

$$\boxed{\Gamma_{ij}^k = \frac{1}{2} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) g^{lk}}$$

Ex \mathbb{R}^n , Euclidean metric. Then $g_{ij} = \delta_{ij}$ so $\Gamma_{ij}^k = 0 \forall i, j, k$.

Geodesics

Def (M, g) Riem mfd, ∇ = Levi-Civita connection.

A curve γ on M is a geodesic if $\frac{D}{dt}(\gamma'(t)) = 0$

i.e. $\gamma'(t)$ is a parallel vector field along γ , " γ has zero acceleration".

Observation: γ = geodesic \Rightarrow

$$\frac{d}{dt} |\gamma'(t)|^2 = \frac{d}{dt} \langle \gamma'(t), \gamma'(t) \rangle = 2 \left\langle \frac{D}{dt} \gamma'(t), \gamma'(t) \right\rangle = 0$$

$\Leftrightarrow |\gamma'(t)|$ is constant. (usually assume non constant mag: $|\gamma'(t)| > 0$)

Ex • Straight lines in \mathbb{R}^n

• for $M \subset \mathbb{R}^{n+k}$: γ = geodesic if tangential component of acceleration $\gamma''(t) \in \mathbb{R}^{n+k}$ is 0.

In coordinates: $\gamma(t) = (x^1(t), \dots, x^n(t))$, $\gamma'(t) = \frac{dx^i}{dt} \partial_i$.
 recall $\frac{Dv}{dt} = \left(\frac{\partial v^k}{\partial t} + \frac{dx^i}{dt} \nabla^i v^k \right) \partial_k$
 $\rightarrow \frac{D}{dt}(\gamma'(t)) = \left(\frac{d^2}{dt^2} x^k + \frac{dx^i}{dt} \frac{dx^j}{dt} \Gamma_{ij}^k \right) \partial_k$

so a geodesic satisfies a second order system

$$\boxed{\frac{d^2}{dt^2} x^k + \frac{dx^i}{dt} \frac{dx^j}{dt} \Gamma_{ij}^k = 0} \quad k=1, \dots, n.$$

Notes 1. This is homogeneity: if $\gamma(t) : (a, b) \rightarrow M$ is a geodesic
 then so is the reparametrization $\tilde{\gamma}(t) = \gamma(ct) : (\frac{a}{c}, \frac{b}{c}) \rightarrow M$.
 (note $\tilde{\gamma}'(0) = c \gamma'(0)$).

2. Can reformulate in terms of the tangent bundle.

U = Coord chart on M , coords x^1, \dots, x^n

$\Rightarrow U \times \mathbb{R}^n$ = coord chart on TM , coords $x^1, \dots, x^n, y^1, \dots, y^n$.

γ = curve in $M \rightsquigarrow \tilde{\gamma}$ = curve in TM given by
 $\tilde{\gamma}(t) = (\gamma(t), \gamma'(t))$.

Then if γ = geodesic, $\tilde{\gamma} = (x^1, \dots, x^n, y^1, \dots, y^n)$ is given by:

$$\left\{ \begin{array}{l} \text{first order} \\ \text{system of} \\ \text{diff eqs} \end{array} \right. \begin{array}{ll} y^i(t) = \frac{dx^i}{dt} & \frac{d}{dt} y^i(t) = -y^i y^j \Gamma_{ij} \\ \vdots & \vdots \\ y^n(t) = \frac{dx^n}{dt} & \frac{d}{dt} y^n(t) = -y^i y^j \Gamma_{ij}^n \end{array}$$

Def The geodesic vector field on TM is the vector field given in
 Coordinates by

$$(y^1, \dots, y^n, -y^i y^j \Gamma_{ij}^1, \dots, -y^i y^j \Gamma_{ij}^n).$$

$$\begin{matrix} 1 & 2 & 3 & 4 \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^n} & \frac{\partial}{\partial y^1} & \frac{\partial}{\partial y^n} \end{matrix}$$

(from the following discussion: coord independent).

Review for 3/11 class:

(M, g) Riem. ^{curve} curve γ is a geodesic if $\frac{D}{dt}(\gamma'(t)) = 0$ wrt. Levi-Civita Connection.

$\rightarrow |\gamma'(t)|$ constant

\rightarrow homogeneous. If $\gamma(t) : (a, b) \rightarrow M$ is a geodesic then so is

$$\tilde{\gamma}(t) = \gamma(ct) : \left(\frac{a}{c}, \frac{b}{c}\right) \rightarrow M. \quad (\text{note } \tilde{\gamma}'(0) = c\gamma'(0))$$

$$\frac{D}{dt}(\tilde{\gamma}'(t)) = \nabla_{\tilde{\gamma}'(t)} \tilde{\gamma}'(t) = \nabla_{c\gamma'(ct)}(c\gamma'(ct)) = c^2 \nabla_{\gamma''(ct)} \gamma'(ct) = 0.$$

\rightarrow Short time existence: for $x \in M$, $v \in T_x M$, small ϵ , $\exists!$ geodesic

$$\gamma = \gamma(t, x, v) : (-\epsilon, \epsilon) \rightarrow M \text{ with } \gamma(0) = x, \gamma'(0) = v.$$

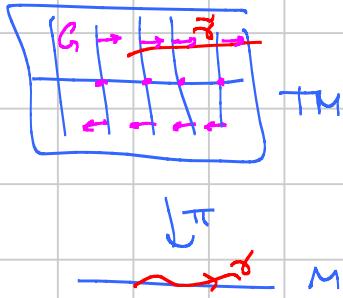
Note: $\gamma(ct, x, cv) = \gamma(t, x, cv)$.

\rightarrow Geodesic flow. Lift any curve γ in M to $\tilde{\gamma}$ in TM by $\tilde{\gamma}(t) = (\gamma(t), \gamma'(t))$,

$\gamma = \pi \circ \tilde{\gamma}$. There is a vector field G on TM ,

the geodesic vector field, so that

$$\gamma \text{ geodesic in } M \Leftrightarrow \tilde{\gamma} = \text{integral curve (flow) of } G.$$



Short time existence of flow of $G \Rightarrow$

given $(x, v) \in TM$, $\exists!$ integral curve $\tilde{\gamma} : (-\epsilon, \epsilon) \rightarrow TM$ with $\tilde{\gamma}(0) = (x, v)$

$\Rightarrow \gamma = \pi \circ \tilde{\gamma}$ is the unique geodesic with $\gamma(0) = x, \gamma'(0) = v$.

The geodesic vector field is constructed so that $\tilde{\gamma}$ is a flow of it.
This implies short time existence of geodesics:

for $(x, v) \in TM$, let $\tilde{\gamma}(t)$ be the time t flow of the geodesic vector field in TM ;

then $\gamma(t) = \pi \circ \tilde{\gamma}(t)$ is a geodesic in M with $\gamma(0) = x, \gamma'(0) = v$,

And conversely. So $\exists!$ geodesic $\gamma: (-\epsilon, \epsilon) \rightarrow M$ with
 $\gamma(0) = x, \gamma'(0) = v$, small ϵ .

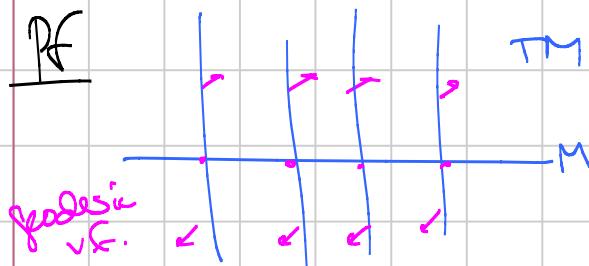
3/9 ↗

Prop $\forall p \in M \quad \exists \epsilon > 0$, nbd U of p , and smooth map

$\gamma: (-\epsilon, \epsilon) \times \Omega \rightarrow M$ where $\Omega = \{(x, v) \mid x \in U, |v| < \epsilon\} \subset TM$

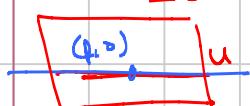
s.t. $t \mapsto \gamma(t, x, v)$ is the unique geodesic γ
with $\gamma(0) = x, \gamma'(0) = v$.

Short time existence:



Can find $\bar{\Omega}, h \setminus d$ of $(p, 0) \in TM$,
such that for $(x, v) \in \bar{\Omega}$, the time t
flow of the geodesic v.f. starting at (x, v)
is defined for $|t| < \delta$: write this flow as

$$t \mapsto \tilde{\gamma}(t, x, v).$$



Might as well assume $\bar{\Omega}$ is of the form

$$\Omega = \{(x, v) \mid x \in U, |v| < \epsilon_0\} \text{ for some } \epsilon_0, \text{ some nbd } U \text{ of } p.$$

Write $\gamma = \pi \circ \tilde{\gamma}$. Then:



$t \mapsto \gamma(t, x, v)$ is a geodesic for $|t| < \delta, |v| < \epsilon_0$
 $\gamma(\frac{t}{c}, x, cv)$ (reparametrization - homogeneity)

$\Rightarrow t \mapsto \gamma(t, x, cv)$ is a geodesic for $|t| < \frac{\delta}{c}, |v| < \epsilon_0$.

Choose $c = \frac{\delta}{2} \Rightarrow$

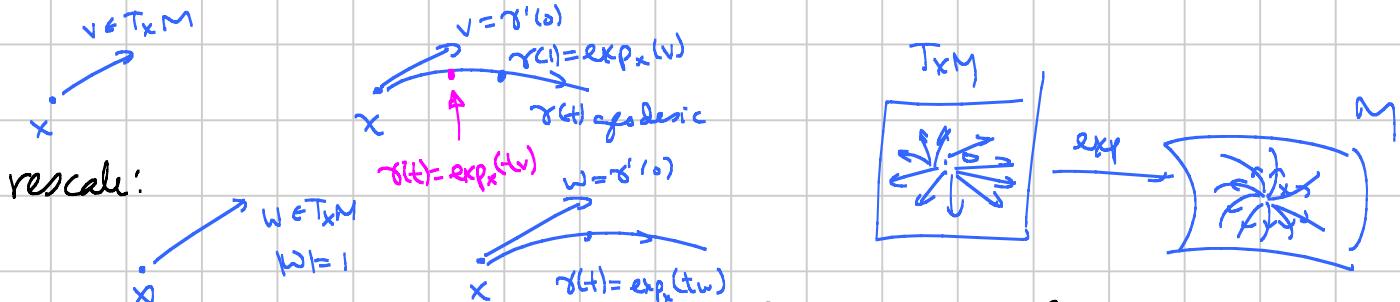
$t \mapsto \gamma(t, x, v)$ is a geodesic for $|t| < 2, |v| < \frac{\epsilon_0 \delta}{2}$. \square

Def Ω as above: $\Omega = \{(x, v) \mid x \in U, |v| < \epsilon\}$

The exponential map $\exp: \Omega \rightarrow M$ is defined by

$$\exp(x, v) = \gamma(1, x, v) = \gamma(|v|, x, \frac{v}{|v|}). \quad x \in U, |v| < \epsilon.$$

For fixed x , get $\exp_x: B_\epsilon(0) \rightarrow M$, $\exp_x(0) = x$.



More generally if $\gamma(t, x, v)$ is defined for $0 \leq t \leq 1 + \delta$ then define $\exp_x(v) = \gamma(1, x, v)$.

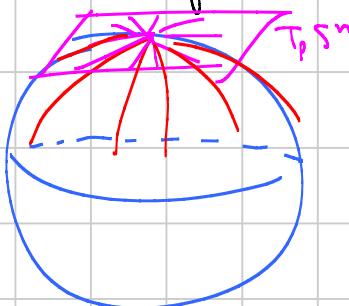
Prop. Fix $x \in M$. $\exists \epsilon > 0$ s.t. $\exp_x: B_\epsilon(0) \rightarrow M$ is a diffeomorphism onto an open subset of M .

$$\begin{aligned} \text{Pf } d(\exp_x)(0)(v) &= \frac{d}{dt} \Big|_{t=0} \exp_x(tv) \\ &= \frac{d}{dt} \Big|_{t=0} \gamma(1, x, tv) \\ &= \frac{d}{dt} \Big|_{t=0} \gamma(1, x, v) \\ &= v \end{aligned}$$

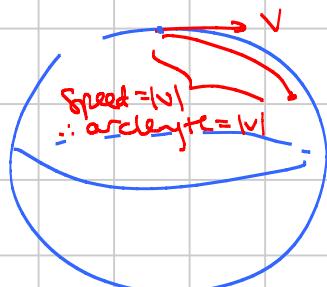
So $d(\exp_x)(0)$ is an isomorphism. Now use Inverse Function Thm. \square

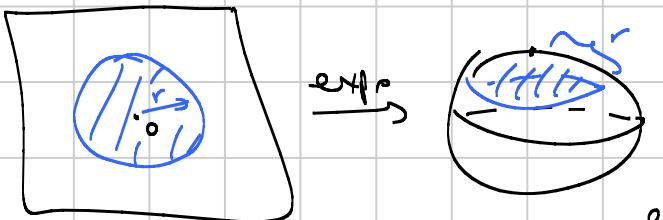
Ex $(S^n, \text{round metric})$.

Geodesics are (arcs of) great circles, parametrized w/ constant speed.



$$\exp_p: \mathbb{R}^n \rightarrow S^{n+1}. \quad \exp_p(v) :$$





$\exp_p : B_r(0) \rightarrow S^r$ is a diffeo
onto its image for $r < \pi$.

$\exp_p(\{v| |v|=\pi\}) = \text{antipodal pt}$

$\exp_p(\{v| |v|=2\pi\}) = p$
etc.

Ex $SO(n)$ with biinvariant metric. By HW, geodesics are 1-parameter subgroups. If $M \in SO(n)$, define

$$\exp(M) = I + M + \frac{1}{2!} M^2 + \frac{1}{3!} M^3 + \dots \in SO(n).$$

Then $\exp(tM)$ is a 1-parameter subgroup \Rightarrow geodesic, and

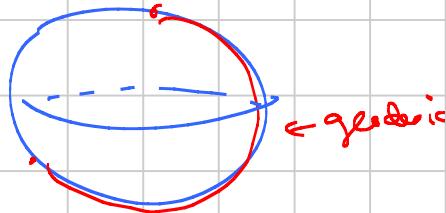
$$\frac{d}{dt}|_{t=0} \exp(tM) = M;$$

$$\begin{aligned} \downarrow \\ \exp(-tM) &= \exp(tM) \\ e \in SO(n), tM \in T_e SO(n) \end{aligned}$$

If $\gamma : [a, b] \rightarrow M$ is a piecewise smooth path in M , define
length $l(\gamma) = \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} dt$. (is particular, continuous)

Claim: if the endpoints p, q of γ are sufficiently close, and γ is a geodesic, then \forall piecewise smooth $\tilde{\gamma}$ with same endpt,
 $l(\tilde{\gamma}) \geq l(\gamma)$.

Note: not true in general:



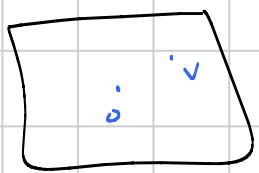
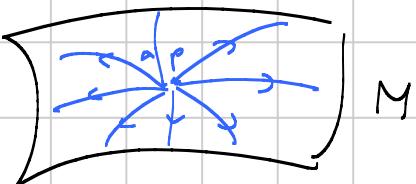
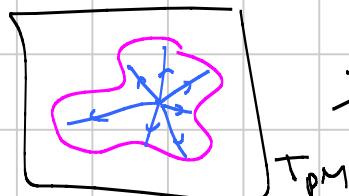
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Review Geodesic = $\gamma(t)$: $\nabla_{\gamma'} \gamma' = 0$.

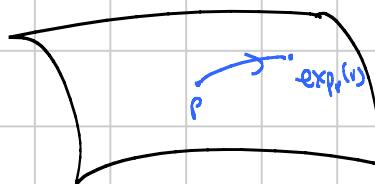
$\exp_p: (\text{Subset of } T_p M) \rightarrow M$

$$\checkmark \quad \mapsto \gamma(1, p, v)$$

where $\gamma(t, p, v)$ = geodesic with
 $\gamma(0) = p, \gamma'(0) = v$.



$$\rightarrow$$



geodesic $\gamma(t, p, v), |\gamma'(t)| = |v|$
 $\text{length} = \int_0^1 \langle \gamma', \gamma' \rangle dt = |v|$.

Note if we fix v , then $\exp(tv) = \gamma(1, p, tv) = \gamma(t, p, v)$

so $\exp(tv)$ is the geodesic with initial conditions $\gamma(0) = p, \gamma'(0) = v$.

Normal Neighborhoods

eventual goal: geodesics are length minimizers.

Def $V = \text{nbhd}$ of $p \in M$ is a normal neighborhood of p if \exists
 $U = \text{nbhd}$ of $0 \in T_p M$ such that $\exp_p: U \rightarrow V$ is a diffeomorphism.

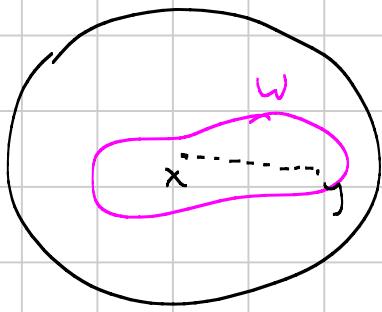
Prop $p \in M$. \exists nbhd W of p and $\epsilon > 0$ such that:

1. $\forall x \in W, \exp_x: B_\epsilon(0) \rightarrow M$ is a diffeo onto its image and

$\exp_x(B_\epsilon(0)) \supset W$ (so W = normal nbhd of each $x \in W$)

2. $\forall x, y \in W, \exists! v \in T_x M$ with $|v| < \epsilon$ st. $y = \exp_x v$;

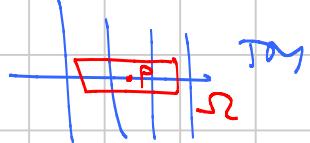
i.e. $\exists!$ geodesic of length $< \epsilon$ joining x and y .



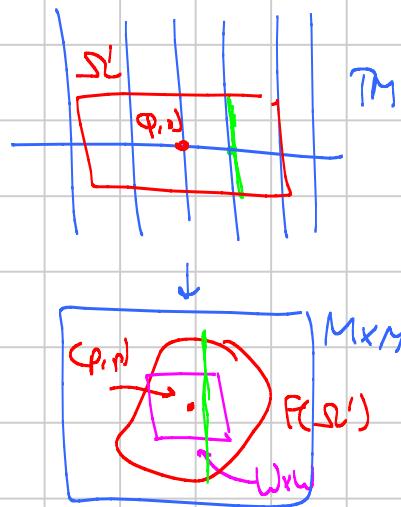
W is called a totally normal neighborhood of p .

Pf $\Omega = \{(x, v) \mid x \in U, |v| < \epsilon\} \subset TM$ as before
($U = \text{hd of } x$).

$\exp: \Omega \rightarrow M$ \rightsquigarrow write $F: \Omega \rightarrow M \times M$
 $(x, v) \mapsto \exp_x v$. $(x, v) \mapsto (x, \exp_x v)$



The $dF(p, 0) = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$ so F is a local diffeo near $(p, 0)$.
Thus \exists nbd $U' \subset U$ of p , $\epsilon' < \epsilon$, s.t. F is a diffeo on
 $\Omega' = \{(x, v) \mid x \in U', |v| < \epsilon'\}$.



There is a nbd W of p with $W \times W \subset F(\Omega')$.

If $x \in W$ then $F((x \times B_{\epsilon}(0))) \supset \{x\} \times W$

so $\exp_x(B_{\epsilon}(0)) \supset W$. This proves #1.

#2 follows directly from #1. \square

Notation: a normal ball with center p and radius r is $\exp_p(B_r(0))$ whose closure lies in a normal nbd V of p :

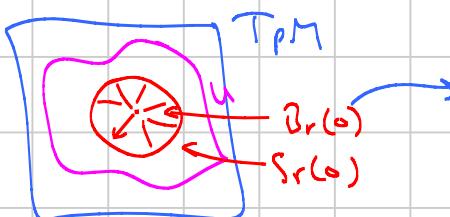
i.e. $\bar{B}_r(0) \subset U$ where $\exp_p: U \xrightarrow[p]{\cong} V$ is a diffeo.
 $\partial B_r(0) = \bar{B}_r(0) - B_r(0)$

Then $\exp_p(S_r(0))$ is called the normal sphere with center p and radius r .

Write

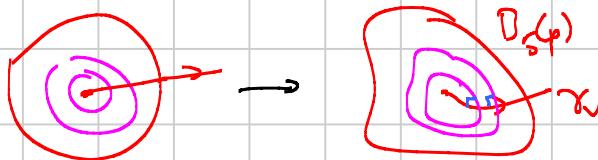
$$B_r(p) = \exp_p(B_r(0))$$

$$S_r(p) = \exp_p(S_r(0))$$



Now: Suppose $B_{r_0}(p)$ is a normal ball.

For $v \in T_p M$ with $|v|=1$, let $\gamma_v: [0, r_0] \rightarrow M$ be the geodesic $\gamma_v(r) = \exp_p(rv)$. ("radial geodesic")

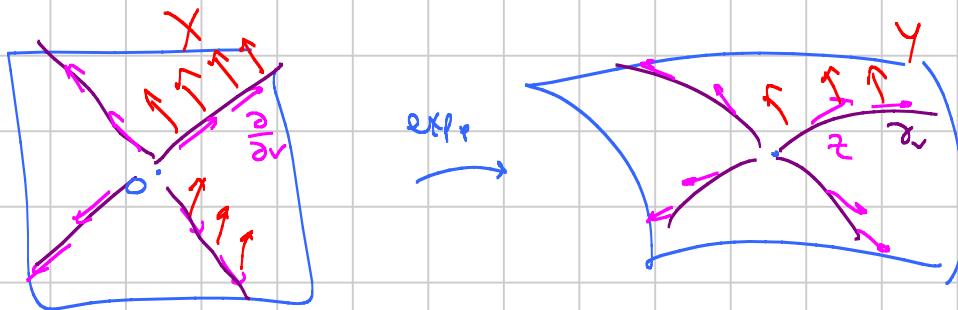


Gauss Lemma γ_v is normal to $S_r(p) \quad \forall r \in (0, r_0]$.

Pf Let $\frac{\partial}{\partial r}$ be the radial vector field on $B_r(0) - \{0\} \subset T_p M$.

Define $Z := (\exp_p)_* \frac{\partial}{\partial r}$. Then

$$Z_{\gamma_v(r)} = \gamma'_v(r). \quad (\text{note } |Z|=1)$$

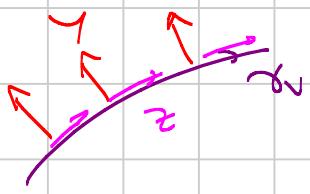


Want: any tangent vector to $S_r(p)$ at $\gamma_v(r)$ is \perp to $Z_{\gamma_v(r)}$.

Let X be any vector field defined on $S_r(0)$; extend this to X on $B_r(0) - \{0\}$ by $X_{rw} = X_w$, $w \in S_r(0)$

It suffices to show that

$$Y := (\exp_p)_* X \quad \text{and} \quad \gamma'_v \quad \text{are orthogonal along } \gamma_v.$$



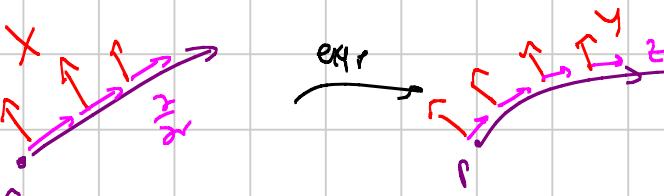
Fix v , and write $f(t) = \langle Y_{\gamma_v(t)}, Z_{\gamma_v(t)} \rangle$. Want $f(t) = 0$.

$$\begin{aligned} \frac{d}{dt} f(t) &= 2 \langle Y, Z \rangle = \langle D_Y Y, Z \rangle + \langle Y, D_Z Z \rangle \\ &= \underbrace{\langle D_Y Z, Z \rangle}_{=0} + \langle [Z, Y], Z \rangle \\ &= 0 \text{ since } 2 \langle D_Y Z, Z \rangle = Y \langle Z, Z \rangle = 0 \end{aligned}$$

$= 0$ since Z is a geodesic

$$[Z, Y] = \left[(\exp_p)_* \frac{\partial}{\partial r}, (\exp_p)_* X \right] = (\exp_p)_* \left[\frac{\partial}{\partial r}, X \right] = 0 \quad (\text{exercise})$$

$\Rightarrow f(t) = c$ for some constant c .



Now we can extend X and $\frac{\partial}{\partial r}$ along the arc $\{tv\}$ continuously to 0, so we can extend $Y = (\exp_p)_* X$ and $Z = (\exp_p)_* \frac{\partial}{\partial r}$ along γ_v to p . From last time,

$(\exp_p)_*(0)$: $T_0(T_p M) \rightarrow T_p M$ is the identity, so $Y_p = X_0$ and $Z_p = \left(\frac{\partial}{\partial r}\right)_0$.

$$\begin{aligned} \Rightarrow c &= \lim_{t \rightarrow 0} \langle Y, Z \rangle_{\gamma_v(t)} = \langle Y, Z \rangle_p \\ &= \langle X, \frac{\partial}{\partial r} \rangle_p \\ &= 0. \quad \square \end{aligned}$$

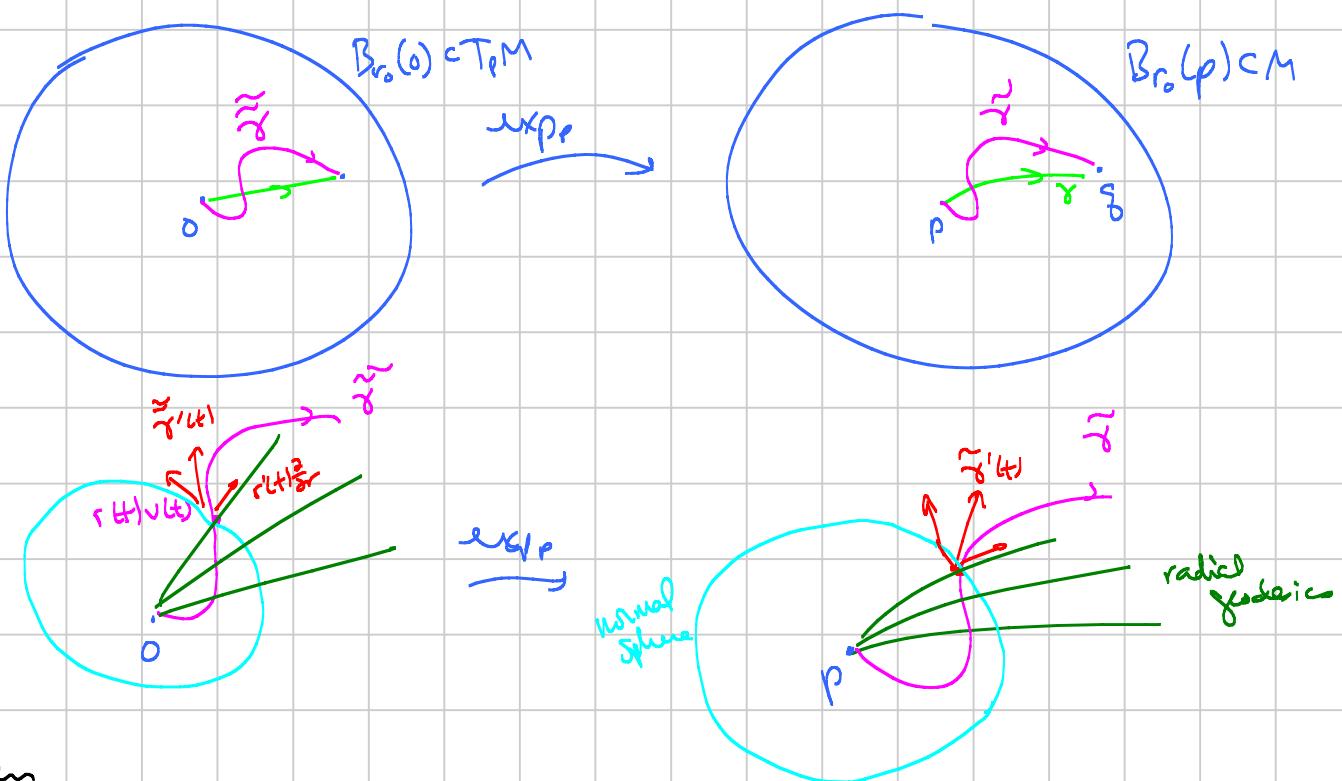
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Prop $p \in M$, $B = B_r(p)$ normal ball centered at p .

$\gamma \in B$, $\gamma: [0, 1] \rightarrow B$ is the geodesic wth $\gamma(0) = p$, $\gamma(1) = \gamma$.
 $(\gamma(t) = \exp_p(t \exp_p^{-1}(\gamma)))$.

If $\tilde{\gamma}: [0, 1] \rightarrow M$ is piecewise smooth wth $\tilde{\gamma}(0) = p$, $\tilde{\gamma}(1) = \gamma$,
then $l(\tilde{\gamma}) \geq l(\gamma)$, wth equality $\Leftrightarrow \tilde{\gamma}$ = reparametrization of γ .

Pf first suppose $\tilde{\gamma}$ lies in B , and define $\tilde{\gamma}: [0, 1] \rightarrow B_{r_0}(0)$ by
 $\exp_p \circ \tilde{\gamma} = \tilde{\gamma}$. Write $\tilde{\gamma}(t) = r(t)v(t)$, $|v(t)| = 1$.



Then

$$\tilde{\gamma}'(t) = (r'(t) \frac{\partial}{\partial r}) + (\text{tangent to sphere})$$

$$\Rightarrow \tilde{\gamma}'(t) = r'(t)z + (\text{tangent to normal sphere})$$

and $z \perp \text{normal sphere}$ by Gauss Lemma

$$\Rightarrow |\tilde{\gamma}'(t)| \geq |r'(t)| |z| = |r'(t)|,$$

so

$$l(\tilde{\gamma}) = \int_0^1 |\tilde{\gamma}'(t)| dt \geq \int_0^1 |r'(t)| dt \geq \int_0^1 r'(t) dt = r(1) = l(\gamma)$$

wth equality \Leftrightarrow no normal component and $r' > 0 \Leftrightarrow \tilde{\gamma} = \gamma$.

(recall $z = (\exp_p)^*(\frac{\partial}{\partial r})$)

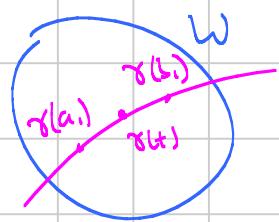
Now: if $\tilde{\gamma}$ doesn't lie in B , let t_0 be first time with $\tilde{\gamma}(t_0) \notin B$. Then

$$l(\tilde{\gamma}) = \int_0^1 |\tilde{\gamma}'(t)| dt \geq \int_{t_0}^1 |\tilde{\gamma}'(t)| dt \geq r_0 > l(\gamma). \quad \square$$

Converse? Are length minimizing geodesics?

Prop $\gamma: [a, b] \rightarrow M$ piecewise smooth, constant speed. If $l(\gamma) \leq l(\tilde{\gamma})$ for any $\tilde{\gamma}$ with same endpoints, then γ is a geodesic.

If $t \in [a, b]$, W = totally normal ball of $\gamma(t)$; so γ maps $[a, b]$ to W for some $a < t < b$. Then $\gamma(a) \rightarrow \gamma(b)$ is a curve in a normal ball. If $\tilde{\gamma}$ = geodesic joining $\gamma(a)$ to $\gamma(b)$, then $l(\tilde{\gamma}) \leq l(\gamma) \Rightarrow l(\tilde{\gamma}) = l(\gamma) \Rightarrow \tilde{\gamma} = \gamma$ up to reparametrization $\Rightarrow \gamma|_{[a, b]}$ is a geodesic.



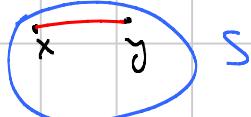
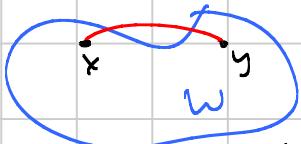
This is true for all t . \square

Geodesic Convexity

See do Carmo, ch 3 sec 4

Recall: any pt has a totally normal neighborhood: $W \ni p$ and $\epsilon > 0$ such that

\cancel{p} any $x, y \in W$ can be connected by a geodesic of length $< \epsilon$: but this could go outside W .



Def A subset $S \subset M$ is (geodesically) convex if $\forall x, y \in S, \exists!$ length minimizing geodesic γ between x and y such that the interior of $\gamma \subset S$.

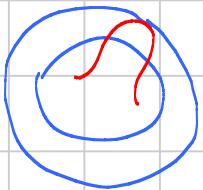
Prop $p \in M$. $\exists \epsilon > 0$ st. any normal ball $B_r(p)$ w/ $r < \epsilon$ is geodesically convex.

Idea: Lemma For any suff.-small ϵ , any geodesic tangent to $S_\epsilon(p)$ lies outside $B_\epsilon(p)$.



Then: connect x, y by a geodesic.

If this strays outside $B_\epsilon(p)$, then it's tangent to $B_r(p)$ for some r but lies inside.



Geodesics and Topology

Def M Riem. The distance between two points $p, q \in M$ is

$$d(p, q) = \inf l(\gamma) \quad \text{over all piecewise smooth } \gamma \text{ from } p \text{ to } q.$$

(note: if γ achieves inf then γ is a geodesic).

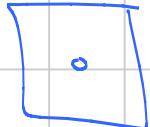
Prop (M, d) is a metric space, and the topology on M agrees with the metric topology.

Pf Metric ball " $B_r(p)$ " = $\{q \mid d(p, q) < r\}$. Claim this agrees with normal ball $B_r(p)$. Recall if $q \in B_r(p)$ then $d(p, q) = l(\gamma) < r$ where γ = geodesic from p to q .

Conversely if $q \notin B_r(p)$ then any path from p to q has length $\geq r \Rightarrow d(p, q) \geq r$. \square

When is there always a minimal geodesic between two points?

not here →

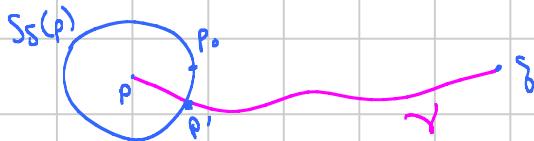


Def M is geodesically complete if all geodesics can be extended to have domain \mathbb{R} : i.e., $\forall p \in M$, \exp_p is defined on all of $T_p M$.

Thm M connected, $p \in M$. If \exp_p is defined on all of $T_p M$, then $\forall q \in M \exists$ geodesic γ joining p to q with $l(\gamma) = d(p, q)$.

Lemma $p, q \in M$. For suff small δ , $\exists p_0 \in S_\delta(p)$ with $d(p, p_0) + d(p_0, q) = d(p, q)$.

Pf Choose δ such that $B_\delta(p) =$ normal ball, and choose $p_0 \in S_\delta(p)$ minimizing $d(p_0, q)$ (\exists since $d(\cdot, q) : S_\delta(p) \rightarrow \mathbb{R}$ is continuous).



If γ joins p to q then if $p' \in \gamma \cap S_\delta(p)$, $b \circ K = \delta$

$$l(\gamma) \geq d(p, p') + d(p', q) \geq d(p, p_0) + d(p_0, q)$$

so $d(p, q) \geq d(p, p_0) + d(p_0, q) \geq d(p, q)$. \square

Pf of Thm. Suppose $d(p, q) = r$. Choose δ, p_0 as in lemma

and write $p_0 = \exp_p(sv)$, $|v| = 1$.

Let γ be the geodesic $\gamma(t) = \exp_p(tv)$. Claim: $\gamma(r) = q$.

Define $I = \{t \in [0, r] \mid d(\gamma(t), q) = r - t\}$. Note $\delta \in I$:

$$d(\gamma(\delta), q) = d(p_0, q) = d(p, q) - \delta = r - \delta.$$

Also I is closed $\Rightarrow \cup \gamma^{-1}(I) = T = \max I$. $T \in I$, $\delta \leq T \leq r$.

If $T = r$ then $d(\gamma(t), g) = 0 \Rightarrow$ done.

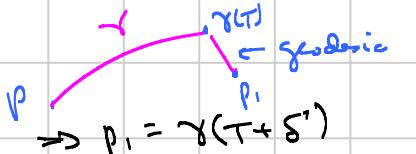
If $T < r$ then apply lemma to $\gamma(T), g$.

$\Rightarrow \exists \delta', p_1 \in S_{\delta'}(\gamma(T))$ with

$$d(\gamma(T), p_1) = \delta', \quad d(p_1, g) = r - T - \delta' \quad (d(\gamma(T), g) = r - T)$$

$$\Rightarrow r = d(p_1, g) \leq d(p_1, p) + d(p, g)$$

$$\Rightarrow d(p, p_1) \geq r - (r - T - \delta') = T + \delta'.$$

Now the path  has length $T + \delta' \rightarrow$ it's a geodesic
 $\Rightarrow p_1 = \gamma(T + \delta')$ $\Rightarrow T$ isn't maximal. \square

→ Sh 8

Theorem (Hopf-Rinow) (M, g) connected Riem. TFAE:

1. (M, g) is geodesically complete, i.e. $\forall p, \exp_p$ is defined on all of $T_p M$
2. for some p, \exp_p is defined on all of $T_p M$
3. all closed bounded (w.r.t. d) subsets of M are compact
4. (M, d) is complete as a metric space.

Moreover (by previous thm), any of those imply that $\forall p, g \in M, \exists$ geodesic γ between p and g s.t. $l(\gamma) = d(p, g)$.

Pf. $1 \Rightarrow 2$ obvious.

2 \Rightarrow 3: $K \subset M$ closed, bounded. Then $\exists R$ with $K \subset B_R^d(p) =$ ball centered at p with radius R in d metric.

$\Rightarrow K \subset \exp_p(B_R(o))$: $\forall g \in K, d(p, g) < R$, so by previous thm \exists geodesic of length $< R$ between p, g

$\Rightarrow K \subset \exp_p(\overline{B_R(o)})$ = compact since it's the continuous image of compact

$\Rightarrow K$ is cpt since closed.

3 \Rightarrow 4: $\{p_n\}$ Cauchy \Rightarrow bounded \Rightarrow sits in some compact $\overline{B_R^d(p)}$

\Rightarrow has a convergent subsequence \rightarrow convergence.

4 \Rightarrow 1: Let γ be a geodesic, assume speed = 1. Say its maximal domain is $I \subset \mathbb{R}$. Local existence $\Rightarrow I$ is open.

Claim I is closed. Let $\{t_n\} \subset I$ satisfy $t_n \rightarrow t$; want $t \in I$.

Note $d(\gamma(t_m), \gamma(t_n)) \leq (\text{length of } \gamma \text{ between } t_m \text{ and } t_n) = |t_m - t_n|$.

$\Rightarrow \{\gamma(t_n)\}$ is Cauchy \Rightarrow has a limit point $p \in M$.

Let $W \ni p$ be a totally normal nbhd, and $\epsilon \in \mathbb{R}^+$ st. $W \subset B_\epsilon(x) \forall x \in W$.

Any geodesic γ of speed 1 starting at any pt $x \in W$ is defined at least on $(-\epsilon, \epsilon)$. Choose n st. $\gamma(t_n) \in W$ and $|t - t_n| < \frac{\epsilon}{2}$.

Then γ is defined at least on $(t_n - \epsilon, t_n + \epsilon)$ and thus at t . \square

Cor The following are geodesically complete:

- any compact M
 - any closed submanifold of a geodesically complete mfd (e.g. Euclidean \mathbb{R}^n)
 - homogeneous manifolds (Hm)
-

Curvature

Gauss: defined "Gaussian curvature" for surfaces.

Then can extend to 2D slices of a mfd: "Sectional Curvature."

Modern formulation: Curvature tensor, measures deviation from being flat (isometric to Euclidean space).

Def The curvature tensor of (M, g) is the $(1, 2)$ tensor

$$X, Y, Z \in \text{Vect}(M) \rightarrow R(X, Y)Z \in \text{Vect}(M)$$

defined by

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z$$

where ∇ = Levi-Civita connection.

Caution: many people use the opposite sign convention.

Note on name: fixing X, Y , the map $Z \mapsto R(X, Y)Z$ is a $(1, 1)$ tensor:

think of $R(X, Y)$ as being an endomorphism of $T_p M$, $R(X, Y) : T_p M \rightarrow T_p M$

Check tensor: $R(fX, Y)Z = \underbrace{\nabla_Y (f \nabla_X Z)}_{f \nabla_Y \nabla_X Z + (Yf) \nabla_X Z} - \underbrace{f \nabla_X \nabla_Y Z}_{f \nabla_{[X, Y]} Z - (Xf) \nabla_Y Z} + \nabla_{[fX, Y]} Z$

$$= f R(X, Y)Z ;$$

$$\begin{aligned} R(X, Y)(fZ) &= \nabla_Y \nabla_X (fZ) - \nabla_X \nabla_Y (fZ) + \nabla_{[X, Y]} (fZ) \\ &= f R(X, Y)Z + (Yf) \nabla_X Z - (Xf) \nabla_Y Z \\ &\quad + \nabla_Y ((Xf)Z) - \nabla_X ((Yf)Z) + ([XY]f)Z \\ &\quad \xrightarrow{(YXf)Z + (Kf)\nabla_Y Z} \xrightarrow{(XYf)Z + (Yf)\nabla_X Z} \\ &= f R(X, Y)Z . \end{aligned}$$

Prop (First) Bianchi identity $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$.

PF LHS = $\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z$ $\rightarrow \nabla_{[X, Y]} Z - \nabla_Z [X, Y]$
 $+ \nabla_Z \nabla_Y X - \nabla_Y \nabla_Z X + \nabla_{[Y, Z]} X$
 $+ \nabla_X \nabla_Z Y - \nabla_Z \nabla_X Y + \nabla_{[Z, X]} Y$
 $= [(X, Y), Z] + \text{cyclic permutations} = 0 . \quad \square$

We can turn the $(1,3)$ tensor $R(X,Y)Z$ into a $(0,4)$ tensor by using the metric:

define $R(X,Y,Z,W) := \langle R(X,Y)Z, W \rangle$.
 $X,Y,Z,W \in T_p M \Rightarrow R$

Prop 1. $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$

2. $R(X,Y,Z,W) = -R(Y,X,Z,W)$

3. $R(X,Y,Z,W) = -R(X,Y,W,Z)$

4. $R(X,Y,Z,W) = R(Z,W,X,Y)$

PF. 1. Bianchi; 2. obvious

3. equivalently: $R(X,Y,Z,Z) = 0$.

$$R(X,Y,Z,Z) = \langle \nabla_Y \nabla_X Z, Z \rangle - \langle \nabla_X \nabla_Y Z, Z \rangle + \langle \nabla_{[X,Y]} Z, Z \rangle$$

$$\cancel{Y \langle \nabla_X Z, Z \rangle} - \cancel{\langle \nabla_X Z, \nabla_Y Z \rangle} \quad \cancel{X \langle \nabla_Y Z, Z \rangle} - \cancel{\langle \nabla_Y Z, \nabla_X Z \rangle}$$

$$= \frac{1}{2} Y X \langle Z, Z \rangle - \frac{1}{2} X Y \langle Z, Z \rangle + \frac{1}{2} [X, Y] \langle Z, Z \rangle$$

$$= 0.$$

4. $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$

$$\underline{R(Y,Z,W,X)} + \underline{R(Z,W,Y,X)} + \underline{R(W,Y,Z,X)} = 0$$

$$\underline{R(Z,W,X,Y)} + \underline{R(W,X,Z,Y)} + \underline{R(X,Z,W,Y)} = 0$$

$$+ \underline{R(W,X,Y,Z)} + \underline{R(X,Y,W,Z)} + \underline{R(Y,W,X,Z)} = 0$$

$$2 R(Z,X,Y,W) + 2 R(W,Y,Z,X) = 0$$

$$\Rightarrow R(Z,X,Y,W) = -R(W,Y,Z,X) = R(Y,W,Z,X). \quad \square$$

In Coordinates: Write $R_{ijk\ell} = \langle R(\partial_i, \partial_j) \partial_k, \partial_\ell \rangle$, $R(\partial_i, \partial_j) \partial_k = R_{ij}{}^\ell \partial_\ell$.

The $R_{ijk\ell} = R_{ijk}{}^m g_{m\ell}$, $R_{ij}{}^\ell = R_{ijk}{}^m g^{ml}$

$$\Rightarrow R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0$$

$$2,3,4 \Rightarrow R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij}.$$

Formula for R_{ijk}^l :

$$\begin{aligned}
 R(\partial_i, \partial_j) \partial_k &= \nabla_j \nabla_i \partial_k - \nabla_i \nabla_j \partial_k \\
 &= \nabla_j (\Gamma_{ik}^l \partial_l) - \nabla_i (\Gamma_{jk}^l \partial_l) \\
 &= \Gamma_{ik}^l \Gamma_{jl}^m \partial_m + (\partial_j \Gamma_{ik}^l) \partial_l - \Gamma_{jk}^l \Gamma_{il}^m \partial_m - (\partial_i \Gamma_{jk}^l) \partial_l
 \end{aligned}$$

$$\Rightarrow R_{ijk}^l = \Gamma_{ik}^l \Gamma_{jl}^m - \Gamma_{jk}^l \Gamma_{il}^m + \partial_j \Gamma_{ik}^l - \partial_i \Gamma_{jk}^l$$

Ex. \mathbb{R}^n , Euclidean metric: $R_{ijk}^l = 0 \quad \forall i, j, k, l.$

Sectional Curvature

Idea: if $n=2$ then curvature tensor is determined by one number, R_{1212} . In general, slice by a plane to get a 2-d mfd.

Def $\sigma \subset T_p M$ 2-dimensional subspace. Then the sectional curvature of σ at p is, for any basis X, Y of σ ,

$$K(\sigma) := \frac{R(X, Y, X, Y)}{|X \wedge Y|^2} \quad \text{where } |X \wedge Y|^2 := \det \begin{bmatrix} g(X, X) & g(X, Y) \\ g(Y, X) & g(Y, Y) \end{bmatrix}.$$

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Check if we replace X, Y by another basis, this doesn't change.

Just need to check 3 elementary changes of basis:

$$1. (X, Y) \rightarrow (Y, X) : R(X, Y, X, Y) = R(Y, X, Y, X) \text{ and } |X \wedge Y|^2 = |Y \wedge X|^2$$

$$2. (X, Y) \rightarrow (\lambda X, Y) : R(\lambda X, Y, \lambda X, Y) = \lambda^2 R(X, Y, X, Y) \text{ and } |(\lambda X) \wedge Y|^2 = \lambda^2 |X \wedge Y|^2$$

$$3. (X, Y) \rightarrow (X+Y, Y) : |(X+Y) \wedge Y|^2 = |X \wedge Y|^2 \text{ and } \det \begin{bmatrix} g(X, X) & g(X, Y) \\ g(Y, X) & g(Y, Y) \end{bmatrix}$$

$$\begin{aligned}
 R(X+Y, Y, X+Y, Y) &= R(X, Y, X+Y, Y) + R(Y, Y, X+Y, Y) \\
 &= R(X, Y, X, Y) + R(X, Y, Y, Y) = R(X, Y, X, Y).
 \end{aligned}$$

Prop The sectional curvature at a point determines the curvature tensor.

$$\begin{aligned}
 \text{Pf} \quad & \left. \frac{\partial^2}{\partial \alpha \partial \beta} \right|_{\alpha=\beta=0} \left\{ \begin{array}{l} \text{determined by } \kappa \text{ (plane thru } X+\alpha Z, Y+\beta W) \\ R(X+\alpha Z, Y+\beta W, X+\alpha Z, Y+\beta W) \\ - R(X+\alpha W, Y+\beta Z, X+\alpha W, Y+\beta Z) \end{array} \right\} \\
 & = \underline{R(Z, W, X, Y)} + \underline{R(Z, Y, X, W)} + \underline{R(X, W, Z, Y)} + \underline{R(X, Y, Z, W)} \\
 & \quad - \underline{R(W, Z, X, Y)} - \underline{R(W, Y, X, Z)} - \underline{R(X, Z, W, Y)} - \underline{R(X, Y, W, Z)} \\
 & \quad \quad \quad R(Z, Y, X, W) + R(X, Z, Y, W) \quad \quad \quad -R(X, W, Y, Z) - R(W, Y, X, Z) \\
 & = 4 \underline{R(X, Y, Z, W)} - \underline{R(Y, X, Z, W)} + \underline{R(Y, X, W, Z)} \quad (\text{Bianchi}) \\
 & = 6 R(X, Y, Z, W). \quad \square
 \end{aligned}$$

Def (M, g) has constant sectional curvature if

$$\kappa(\sigma) = \text{Constant } K_0 \quad \forall p \in M \text{ and } \forall \sigma = 2\text{-plane at } p.$$

Ex. $(\mathbb{R}^n, \text{flat})$

(S^n, round)

$(H^n, \text{hyperbolic})$

$(\mathbb{R}^n/\mathbb{Z}^n = T^n, \text{flat})$

more generally M/G , $M = \text{Const sectional curvature}$, $G = \text{gp of isometries}$

in fact this gives all constant sectional curvature.

Prop Constant sectional curvature K_0

$$\Leftrightarrow R(X, Y, Z, W) = K_0 (\langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle).$$

Pf Call $R'(\sigma) = R'(X, Y, Z, W)$. Easy to check: this satisfies all same properties as R . Also "sectional curvature" $\kappa'(X, Y) = \frac{R'(X, Y, X, Y)}{\|X \wedge Y\|^2} = \frac{K_0 (\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle \langle Y, X \rangle)}{\|X \wedge Y\|^2} = K_0$

So since Sectional Curvature determines Riem Curv, $R = R'$. \square

Two more curvatures

so far: $X, Y \in \mathcal{L} \rightarrow R(X, Y) \in \mathcal{L}$

$\sigma \rightarrow K(\sigma)$ sectional curv.

Several approaches.

First, in terms of an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_p M$:

Def $X, Y \in T_p M$. Then

$\{e_i\}$ other books
may have
different
normalizations

$$\text{Ricci tensor } \text{Ric}_p(X, Y) = \frac{1}{n-1} \sum_{i=1}^n R(X, e_i, Y, e_i) \quad (0,2)\text{ tensor}$$

$$\text{Ricci curvature } \text{Ric}_p(X) = \text{Ric}_p(X, X) \quad \text{quadratic form}$$

[note: $\text{Ric}_p(X, Y)$ is a symmetric bilinear form so it's determined by $\text{Ric}_p(X)$]

$$\text{Scalar curvature } S(p) = \frac{1}{n} \sum_{i=1}^n \text{Ric}_p(e_i) \quad (= \frac{1}{n(n-1)} \sum_{i,j} R(e_i, e_j, e_i, e_j)).$$

These are independent of the choice of e_1, \dots, e_n :

Define $\varphi_{X,Y}: T_p M \rightarrow T_p M$ by $\varphi_{X,Y}(Z) = R(X, Z)Y$.

Then

$$\text{Ric}_p(X, Y) = \frac{1}{n-1} \text{Tr } \varphi_{X,Y} \quad \text{indep. of basis}.$$

Now $\text{Ric}_p(X, Y)$ is a bilinear form so $\exists T: T_p M \rightarrow T_p M$ linear s.t.

$$\text{Ric}_p(X, Y) = \langle T(X), Y \rangle.$$

Then

$$S(p) = \frac{1}{n} \text{Tr } T : \text{ note } \text{Ric}_p(e_i, e_i) = \langle T(e_i), e_i \rangle = (i,i) \text{ entry of } T.$$

Local Coordinates: note $\{\partial_i\}$ isn't usually orthonormal.

$$R_{ij} := \text{Ric}_p(\partial_i, \partial_j).$$

The map $\varphi_{\partial_i, \partial_j}: Z \mapsto R(\partial_i, Z)\partial_j$ sends $\partial_k \mapsto R_{ikj}\partial_k$

so the (k, l) entry of $\varphi_{\partial_i, \partial_j}$ is R_{ikj}^l

$$\rightarrow R_{ij} = \frac{1}{n-1} \text{tr } \varphi_{\partial_i, \partial_j} = \boxed{\frac{1}{n-1} R_{ikj}^k = \frac{1}{n-1} R_{ikjl} g^{lk} = R_{ij}}$$

$$R_{ij} = \langle T(\partial_i), \partial_j \rangle = \langle T_i^k \partial_k, \partial_j \rangle = T_i^k g_{kj} \Rightarrow T_i^k = R_{ij} g^{jk}$$

$$\Rightarrow S = \frac{1}{n} \text{Tr } T = \frac{1}{n} T_i^i = \frac{1}{n} R_{ij} g^{ji} = \boxed{\begin{aligned} &= \frac{1}{n(n-1)} R_{ikj} g^{ji} g^{ik} = S \\ &= \frac{1}{n(n-1)} R_{ikjl} g^{ij} g^{kl} \end{aligned}}$$

What are these curvatures?

Weyl: View the curvature tensor algebraically.

Let $V = T_p M$. Recall a $(0,2)$ -tensor is in $V^* \otimes V^*$,

Antisymmetric is in $\Lambda^2 V^*$. The curvature tensor R is in:

$$V^* \otimes V^* \otimes V^* \otimes V^* \rightarrow (\Lambda^2 V^*) \otimes (\Lambda^2 V^*) \rightarrow \text{Sym}^2 \Lambda^2 V^*$$

Define the Bianchi map $b: \text{Sym}^2 \Lambda^2 V^* \rightarrow$
 $b(T)(X, Y + \omega) = T(XY) + T(YX) + T(ZXY).$

P Then $R \in \ker b =: C(V) \subset \text{Sym}^2 \Lambda^2 V^*$

Note $O(n)$ acts on $V, V^*, \text{Sym}^2 \Lambda^2 V^*, C(V)$.

Fact: we can decompose $C(V)$ as an $O(n)$ -module:

$$C(V) = R \oplus \underbrace{\text{Sym}^2 V^*}_{\substack{\text{traceless symm bilinear forms}} \atop \text{all symm bilinear forms}} \oplus W(V)$$

$$\text{all symm bilinear forms} = \text{Sym}^2 V^*$$

where the map $C(V) \rightarrow \text{Sym}^2 V^*$ is $(X, Y) \mapsto \text{tr } T(X, Y, \cdot)$

and the map $\text{Sym}^2 V^* \rightarrow R$ is $T \mapsto \text{tr } T$.

$$\text{So } R \xrightarrow{\quad \text{Ric}(X, Y) \quad} \text{Ric} \xrightarrow{\quad \text{Sym}^2 V^* \quad} \begin{matrix} S(p) \\ \oplus \\ R \end{matrix}$$

What about $W(V)$? The component of R in $W(V)$ is the Weyl tensor of R . Facts:

- if $n \geq 5$, $W(V)$ is irreducible
- if $n \leq 3$, $W(V) = 0$. (R is determined by Ric!)

Review:

$$\text{(1,3) curvature tensor } R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

$$\text{(0,4) curvature tensor } R(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle$$

$$\text{Sectional curvature } \sigma \subset T_p M \text{ 2-plane, } e_1, e_2 \text{ ONB for } \sigma \\ \Rightarrow K(\sigma) = R(e_1, e_2, e_1, e_2)$$

$$\text{Ricci tensor } \text{Ric}_p(X, Y) = \frac{1}{n-1} \sum_{i=1}^n R(X, e_i, Y, e_i) \quad \{e_1, \dots, e_n\} \text{ ONB for } T_p M$$

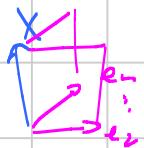
$$\text{Ricci curvature } \text{Ric}_p(X) = \text{Ric}_p(X, X)$$

$$\text{Scalar curvature } S(p) = \frac{1}{n} \sum_{i=1}^n \text{Ric}_p(e_i).$$

If X = unit vector in $T_p M$, then

$\text{Ric}_p(X)$ = average sectional curvature of planes through X .

Complete X to an ONB X, e_2, \dots, e_n , and let $\sigma_i = \langle X, e_i \rangle$.



$$K(\sigma_i) = R(X, e_i; X, e_i),$$

$$\text{Ric}_p(X) = \frac{1}{n-1} \sum_{i=2}^n K(\sigma_i) \quad (\text{note } R(X, X, X, X) = 0).$$

Nice computation: parametrize planes through X by

$V \in \{\text{unit vectors } \perp \text{ to } X\} = S^{n-2}$. With the usual measure on S^{n-2} ,

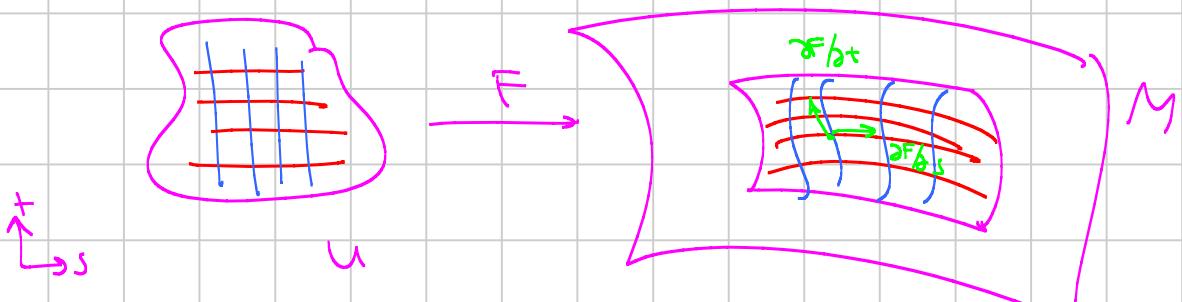
the average $K(\sigma_{X,V})$ over all V is $\text{Ric}_p X$.

In particular: for a space of constant sectional curvature K_0 ,

$$\text{Ric}_p(X) = K_0 |X|^2, \quad \text{Ric}_p(X, Y) = K_0 \langle X, Y \rangle, \quad S = K_0.$$

Derivatives on Surfaces

Def A parametrized surface is a smooth map $F: U \rightarrow M$.



$$\text{Write } \frac{\partial F}{\partial s} = F_*(\frac{\partial}{\partial s}), \quad \frac{\partial F}{\partial t} = F_*(\frac{\partial}{\partial t}).$$

Note $F(s_0, \cdot)$, $F(\cdot, t_0)$ are curves in M for fixed s_0, t_0 .

A vector field V along the surface is in particular a vector field along each of these curves: Can define " $\frac{D}{dt}$ " for V along these curves.

$$F(\cdot, t_0) \xrightarrow{\frac{D}{ds} V} \quad F(s_0, \cdot) \xrightarrow{\frac{D}{dt} V}.$$

→ get vector fields $\frac{D}{ds} V, \frac{D}{dt} V$ on the surface.

Prop $\frac{D}{ds} \frac{\partial F}{\partial t} = \frac{D}{dt} \frac{\partial F}{\partial s}$.

Pf. $\nabla_{\frac{\partial F}{\partial s}} \frac{\partial F}{\partial t} - \nabla_{\frac{\partial F}{\partial t}} \frac{\partial F}{\partial s} = \left[\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t} \right] = F_* \left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right] = 0.$

Prop $\frac{D}{dt} \frac{D}{ds} V - \frac{D}{ds} \frac{D}{dt} V = R \left(\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t} \right) V.$

Pf More or less by definition: $\frac{D}{ds} V = \nabla_{\frac{\partial F}{\partial s}} V, \quad \frac{D}{dt} V = \nabla_{\frac{\partial F}{\partial t}} V$
and $\left[\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t} \right] = F_* \left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right] = 0. \quad \square$

Jacobi vector fields

Let $\gamma = \text{geodesic}$.

Say we have a 1-parameter family of geodesics $\gamma_s(t)$, $s \in (-\epsilon, \epsilon)$,

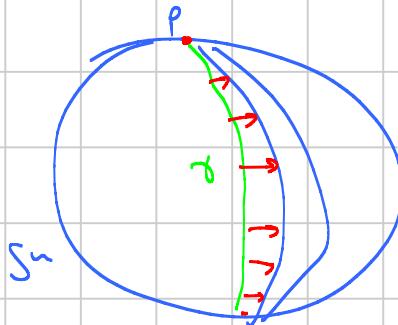
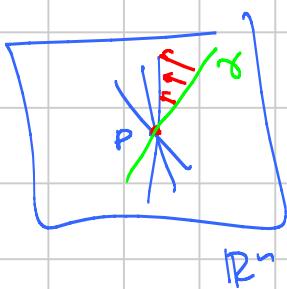
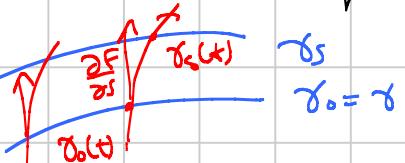
Smoothly varying, $\gamma_0 = \gamma$. Define $F(s, t) = \gamma_s(t)$.

The infinitesimal change in geodesic is $\frac{\partial}{\partial s} F(s, t)$: at $s=0$ this is

a vector field along γ_0 . Call this

$$J(t) := \frac{\partial F}{\partial s}(0, t).$$

This measures the "spread" of the geodesics.



The difference between these is curvature.

Differential equation for $J(t)$:

Fixing $s = s_0 \Rightarrow F(s_0, \cdot)$ is a geodesic $\Rightarrow \frac{D}{dt} \frac{\partial F}{\partial t} = 0$

$$\begin{aligned} 0 &= \frac{D}{ds} \frac{D}{dt} \frac{\partial F}{\partial t} \\ &= \frac{D}{dt} \frac{D}{ds} \frac{\partial F}{\partial t} - R\left(\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right) \frac{\partial F}{\partial t} \\ &= \frac{D}{dt} \frac{D}{ds} \frac{\partial F}{\partial s} + R\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s}\right) \frac{\partial F}{\partial t}. \end{aligned}$$

Plug in $s = 0$:

$$0 = \frac{d^2}{dt^2} J(t) + R(\gamma', J) \gamma' \quad (*)$$



Def A vector field J along a geodesic $\gamma: [0, a] \rightarrow M$ is a Jacobi field if (*) holds.

So. if $F(s, t) = \gamma_s(t)$ is a family of geodesics (no assumption on $\gamma_s(0)$)
then $\frac{\partial F}{\partial s}(0, t) = J(t) \in$ Jacobi field along $\gamma_0 = \gamma$.

Prop 3) Jacobi field for sufficiently small a and specified initial conditions

$$J(0), J'(0) \in T_p M.$$

$$\frac{D}{dt} J(0)$$

PF Choose ONB e_1, \dots, e_n of $T_p M$ \rightsquigarrow by parallel transport, get $e_i(t), \dots, e_n(t)$ ONB along γ . Want to find

$$\begin{aligned} J(t) &= (\varepsilon) f^i(t) e_i(t) \\ \Rightarrow \frac{DJ}{dt} &= \frac{df^i}{dt} e_i(t) \quad \text{since } e_i(t) \text{ parallel} \\ \Rightarrow \frac{D^2 J}{dt^2} &= \frac{d^2 f^i}{dt^2} e_i(t). \end{aligned}$$

Write $r_{ij}(t) = \langle R(\gamma'(t), e_i(t)) \gamma'(t), e_j(t) \rangle$. Then
 $(*)$ holds $\Leftrightarrow \frac{d^2 f^i}{dt^2} e_i(t) + f^j R(\gamma', e_j) \gamma' = 0$
 $\Leftrightarrow \frac{d^2 f^i}{dt^2} + f^j(t) r_{ij}(t) = 0$.

This is a 2nd order linear system of ODEs. \square

Cor The vector space of Jacobi v.f.s along a geodesic is $2n$ -dimensional.

Two "trivial" Jacobi v.f.s:

$$\cdot J_1(t) = \gamma'(t) \quad \frac{D}{dt} \gamma' = 0 \quad J_1(0) = \gamma'(0), J_1'(0) = 0$$

this corresponds to the variation $F(s,t) = \gamma_s(t) = \gamma(t+s)$

$$\cdot J_2(t) = t \gamma'(t) \quad \frac{D^2}{dt^2} (t \gamma'(t)) = 0 \quad J_2(0) = 0, J_2'(0) = \gamma'(0)$$

this corresponds to $F(s,t) = \gamma_s(t) = \gamma((s+1)t)$.

Note: $\{ \text{Jacobi along } \gamma(t) \} \longrightarrow \mathbb{R}^2$

$$J(t) \longmapsto (\langle J(0), \gamma'(0) \rangle, \langle J'(0), \gamma'(0) \rangle)$$

$\text{this is a surjective linear map because of } J_1, J_2$. The kernel is
 $\{ \text{Jacobi with } J(0), J'(0) \perp \gamma'(0) \} = (2n-2)\text{-diml vector space}$.

PF $\langle J(t), \gamma'(t) \rangle = \langle J'(0), \gamma'(0) \rangle t + \langle J(0), \gamma'(0) \rangle$

$\Leftrightarrow J(0), J'(0) \perp \gamma'(0) \Leftrightarrow J(t) \perp \gamma'(t) \forall t.$

PF Write $|\gamma'(t)| = a$. Then $e_i(t) = \frac{\gamma'(t)}{a}$.

$\cancel{\text{P}}$ $\langle J(t), \gamma'(t) \rangle = a f'(t)$, $r_{ij}(t) = \langle R(\gamma', e_i) \gamma', e_j \rangle = 0$
 $\Rightarrow \frac{d^2 f'}{dt^2} = 0 \Rightarrow \langle J(t), \gamma'(t) \rangle = At + B$.
 $t=0 \Rightarrow B = \langle J(0), \gamma'(0) \rangle$; $A = \frac{d}{dt}|_{t=0} \langle J, \gamma' \rangle = \langle J', \gamma' \rangle(0)$. D

Break up $\{ \text{Jacobi v.f.} \}$ into subspaces:

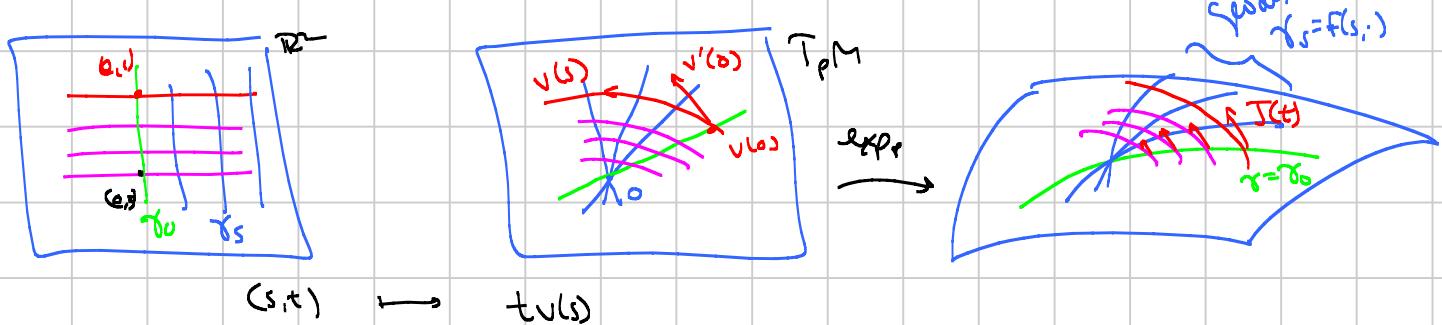
dim	no restriction	J normal to γ
$J(0)$ arbitrary	$2n$	$2n-2$ ← cut out J_1, J_2
$J(0)=0$	n	$n-1$ ← cut out J_2

What are these?

satisfies $F(s, 0) = \gamma$.

let $\gamma_s(t)$ be a family of geodesics wth $\gamma_s(0) = \rho$ $\forall s: F(s, t) := \gamma_s(t)$

Each γ_s is determined by $\gamma'_s(0) = v(s)$: $F(s, t) = \gamma_s(t) = \exp_p(tv(s))$.



Then $J(t) = \frac{\partial F}{\partial s}(0, t) = (d(\exp_p))_{tv}(tw)$ where $v = v(0) = \gamma'(0)$



For $F(s,t) = \exp_p(t\gamma(s))$, $J(t)$ is a Jacobi vector field along γ_0 .

What are its initial conditions?

$J(0) = 0$ since $F(s,0) = p$. Also:

$$J(t) = (\exp_p)_t v(t\omega) \quad \text{where } \omega = \gamma'(0)$$

$$\Rightarrow \frac{DJ}{dt}(0) = \left. \frac{D}{dt} \right|_{t=0} (t(\exp_p)_t v(\omega)) = (\exp_p)_0(\omega) = \omega.$$

$$\text{So: } J(0) = 0, J'(0) = \frac{DJ}{dt}(0) = \omega.$$

#6 ↗

Conversely:

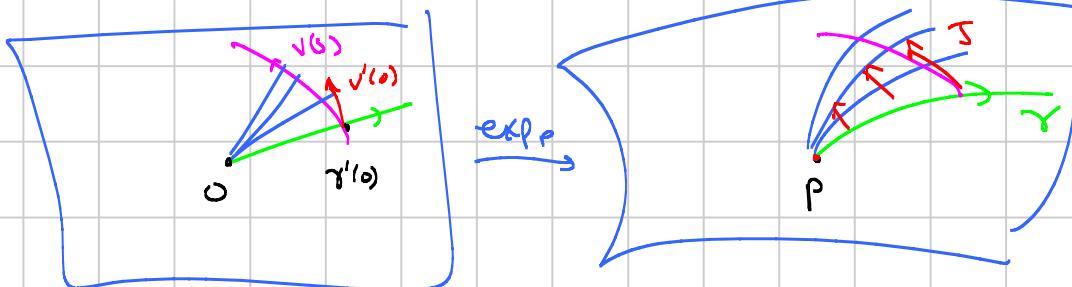
Prop $\gamma: [0,a] \rightarrow M$ geodesic, J = Jacobi vector field along γ with $J(0) = 0$.

Then \exists family of geodesics $F(s,t) = \gamma_s(t)$ with $\gamma_0 = \gamma$

$$\text{st. } J(t) = \frac{\partial F}{\partial s}(0, t).$$

actually holds without this too.

Pf Try $F(s,t) = \exp_p(t\gamma(s))$. Choose path $\{\gamma(s)\} \subset T_p M$ with $\gamma(0) = \gamma'(0)$, $\gamma'(0) = J'(0)$. Then $\frac{\partial F}{\partial s}(0, t) =: \bar{J}(t)$ is a Jacobi vector field, and J, \bar{J} have same initial condition
so $J = \bar{J}$. \square



Now: $F(s, t) = \exp_s(tv(s))$, $J(t) = \frac{\partial F}{\partial s}(0, t)$, $v(0) = v = \gamma'(0)$
 $J(0) = 0$, $J'(0) = w = v'(0)$. Assume $|w| = 1$.

Prop With these assumptions:

$$|J(t)|^2 = t^2 - \frac{1}{3} R(v, w, v, w) t^4 + o(t^4)$$

$$\lim_{t \rightarrow 0} \frac{o(t^4)}{t^4} = 0.$$

Pf $J(0) = 0$, $J'(0) = w$, $J''(0) = -(R(\gamma', J)\gamma')(0) = 0$.

$$\langle J, J \rangle(0) = 0$$

$$\langle J, J \rangle'(0) = 2 \langle J(0), J'(0) \rangle = 0$$

$$\langle J, J \rangle''(0) = 2 \langle J, J'' \rangle(0) \Rightarrow \langle J', J' \rangle(0) = 2$$

$$\langle J, J \rangle'''(0) = 2 \langle J, J''' \rangle(0) + 6 \langle J', J'' \rangle(0) = 0$$

$$\langle J, J \rangle^{(4)}(0) = 2 \cancel{\langle J, J^{(4)} \rangle(0)} + 8 \langle J', J'' \rangle(0) + 6 \cancel{\langle J'', J'' \rangle(0)}$$

To calculate $J'''(0)$, note for any vector field w along γ :

$$\langle R(\gamma', J)\gamma', w \rangle = \langle R(\gamma', w)\gamma', J \rangle$$

$$\left. \frac{d}{dt} \right|_{t=0} \langle \underbrace{\frac{D}{dt} R(\gamma', J)\gamma'}_{= J''}, w \rangle = \langle R(\gamma', w)\gamma', J' \rangle(0) = \langle R(\gamma', J')\gamma', w \rangle(0)$$

$$- \langle J'''(0), w(0) \rangle$$

$$\rightarrow J'''(0) = -R(\gamma', J')\gamma'(0) = -R(v, w)v$$

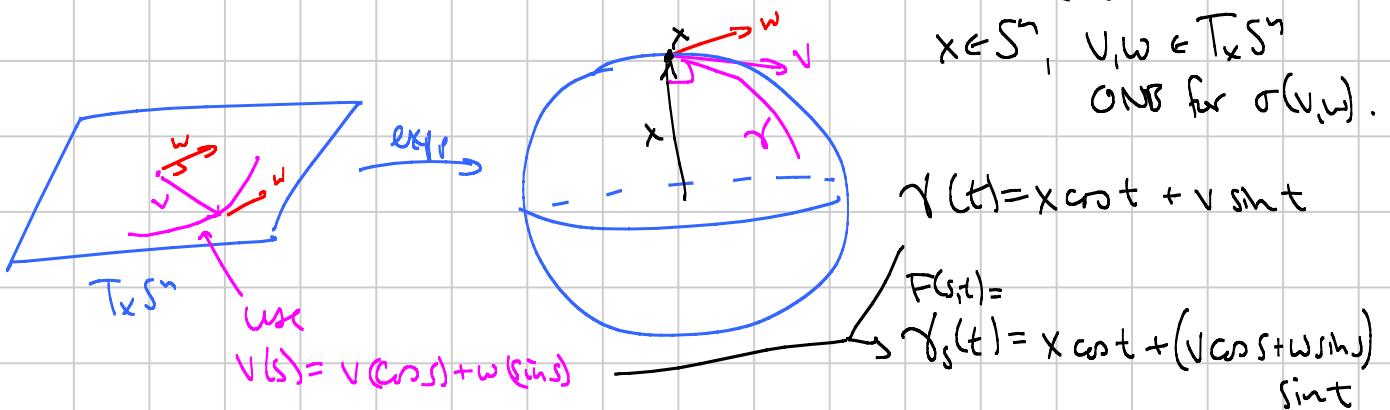
$$\Rightarrow \langle J, J \rangle^{(4)}(0) = 8 \langle J', J'' \rangle(0) = -8 R(v, w, v, w). \quad \square$$

Cor If $|v| = 1$ (γ par. by arc length), $|w| = 1$, $\langle v, w \rangle = 0$,

and σ = plane grid by v, w , then

$$|J(t)|^2 = t^2 - \frac{1}{3} K(\sigma) t^4 + o(t^4).$$

Can use this to calculate sectional curvature for $(S^n, \text{round metric})$.



$$\Rightarrow \bar{J}(t) = \frac{\partial F}{\partial s}(0, t) = w \sin t$$

$$\Rightarrow |\bar{J}(t)|^2 = \sin^2 t = \left(t - \frac{t^3}{3} + \dots\right)^2 = t^2 - \frac{t^4}{3} + \dots$$

$$\rightarrow K(\sigma(v, w)) = 1.$$

(Similarly for sphere of radius R : $K(\sigma(v, w)) = \frac{1}{R^2}$.)

Conjugate Points

Def $\gamma: [a, b] \rightarrow M$ geodesic. Two points $p = \gamma(t_0), q = \gamma(t_1)$ along γ are conjugate if \exists nonzero Jacobi field J along γ w.t. $J(t_0) = J(t_1) = 0$.

One way to get this: γ = part of a family of geodesics between p and q .



The dimension of the vector space $\{J(t_0) = J(t_1) = 0, J \text{ Jacobi}\}$ is the multiplicity of the conjugate point (think: k-dim family of geodesics).

Ex. (S^n , round): Antipodal pts are conjugate along any geodesic, with multiplicity $(n-1)$.

Def: g is the first conjugate to p along γ if no other conjugate points before g . The Conjugate locus of p is $\{\text{first conjugates}\}$ over all geodesics.

Ex: S^n : Conjugate locus of p is $\{-p\}$

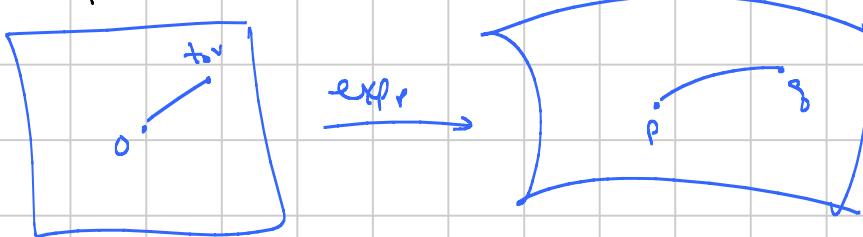
2-d ellipsoid:
(more generic)



\mathbb{R}^n : no conjugate locus (Jacobi fields satisfy $J''(t)=0$)

Prop $\gamma: [0, \bar{a}] \rightarrow M$ geodesic, $\gamma(0) = p$, $\gamma(t) = \exp_p(tv)$, $v = \gamma'(0)$.

Then $g = \gamma(t_0) = \exp_p(t_0v)$ is conjugate to p along $\gamma \Leftrightarrow t_0v$ is a critical point of \exp_p , i.e. $d(\exp_p)_{t_0v}: T_p M \rightarrow T_g M$ is not surjective.



Pf g conjugate $\Leftrightarrow \exists$ nonzero Jacobi field $J(t)$ with $J(0) = J(t_0) = 0$.

Recall if $J'(0) = w \neq 0$ then $J(t) = d(\exp_p)_{t_0v}(tw)$, so

$\exists J^{t_0} \in J(t_0) = 0 \Leftrightarrow \exists w \neq 0$ with $d(\exp_p)_{t_0v}(w) = 0 \Leftrightarrow \ker d(\exp_p)_{t_0v} \neq 0$. \square

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2

Hadamard (Hadamard-Cartan) Thm

by topf-know,
exp is defined on all of $T_p M$

M Complete Riemannian with nonpositive sectional curvature:

$$K(\sigma) \leq 0 \quad \forall \sigma \in T_p M \quad \forall p.$$

Then $\forall p \in M$, $\exp_p: \tilde{T}_p M \xrightarrow{\sim} M$ is a Covering map.

If M is simply connected, then $\exp: \tilde{R}^n \rightarrow M$ is a diffeomorphism.

Ex: R^n, H^n ; not S^n (Cor: S^n can't have a metric of ≤ 0 sect. curv.)

Lemma 1 M (geod.) complete Riem, nonpositive sectional curvature.

(1) Conjugate locus (p) = $\emptyset \forall p \in M$: no conjugate pts to p along any geodesic.

(2) \exp_p is a local diffeo.

Pf (1) $J =$ Jacobi field along γ , $\gamma(0) = p$, $\gamma(t_0) = q$, $J(0) = J(t_0) = 0$.

$$\frac{d}{dt} \langle J(t), J(t) \rangle = 2 \langle J'(t), J(t) \rangle$$

$$\frac{d^2}{dt^2} \langle J(t), J(t) \rangle = 2 \langle J''(t), J(t) \rangle + 2 \langle J'(t), J'(t) \rangle$$

$$= 2 |J'|^2 - 2 \langle Q(\gamma'(t), J(t)) \gamma'(t), J(t) \rangle$$

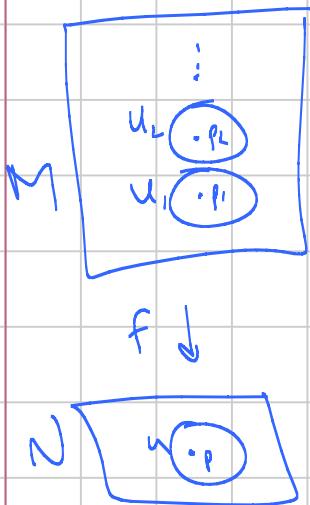
$$= 2 |J'|^2 - 2 K(p) \underbrace{|J' \wedge J|^2}_{\text{sum by } \gamma, J} \geq 0$$

But if J has two zeros then $|J|^2$ has a max, where $\frac{d^2}{dt^2} |J|^2 < 0$.

(2) Follows from previous prop. \square

Lemma 2 M, N Riem, M geod. complete, $f: M \rightarrow N$ surjective local isometry (in particular, local diffeo). Then f is a covering map.

Pf $p \in N$, $f^{-1}(p) = \{p_i\}$. Let $B_r(p) = \text{normal ball in } N$, $\exp_p: B_r(o) \rightarrow B_r(p)$ differs.
Write $U = B_r(p)$, $U_i = \exp_{p_i}(B_r(o)) \subset M$ (\exists since complete).



Claim: $f^{-1}(U) = \coprod U_i$, $f: U_i \rightarrow U$ diffeo.

① $f(U_i) \subset U$: $g \in U_i \Rightarrow \exists$ geodesic γ from p_i to g , $l(\gamma) < r$.
Local isometry $\Rightarrow f \circ \gamma$ = geodesic from p to $f(g)$,
 $l(f \circ \gamma) < r \Rightarrow f(g) \in U$.

② $f: U_i \rightarrow U$ diffeo.

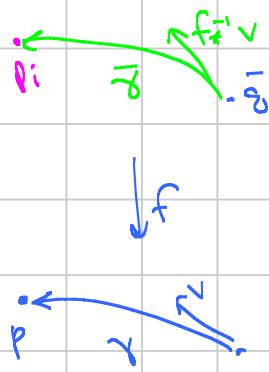
$$\begin{array}{ccc}
 B_r(o) & \xrightarrow{\exp_p:} & U_i \\
 f_* \downarrow \approx & & f \downarrow \\
 B_r(o) & \xrightarrow{\exp_p} & U
 \end{array}
 \quad \text{Diagram commutes:}$$

$\xrightarrow{\quad \quad \quad}$ \xrightarrow{f} $\xrightarrow{f \circ v}$

 p_i $\quad \quad \quad$ p

Since f_* , \exp_p are diffeos, \exp_p is injective \Rightarrow bijective,
 $\Rightarrow f$ is bijective $\Rightarrow \exp_{p_i}, f$ are diffeos.

③ Pf of claim: $f^{-1}(U) \stackrel{?}{=} \coprod U_i$. Suppose $\bar{q} \in f^{-1}(U)$, $q = f(\bar{q})$.



Reverse geod. $q \rightarrow g$ to get geod. γ from g to p .

Write $v = \gamma'(o) \Rightarrow \exists$ geodesic $\bar{\gamma}$ with $\bar{\gamma}(o) = \bar{q}$

$\bar{\gamma}'(o) = (f_*)^{-1}(v)$. Then $f \circ \bar{\gamma} = \gamma$ so

the endpt of $\bar{\gamma}$ is some $p_i \Rightarrow \bar{q} \in U_i$.

If \exists 2 geods $\bar{\gamma}_1, \bar{\gamma}_2$ from \bar{q} to p_1, p_2 then they must project to the same geodesic from g to p by uniqueness. \Rightarrow D

Pf of Hadamard $\exp_p: T_p M \rightarrow M$ well-defined, surjective.

Lemma 1 \Rightarrow local diffeo. So can pull back metric on M to
metric on $T_p M$; then \exp_p is a local isometry.

Now straight lines $\{tv\}$ through $0 \in T_p M$ are geodesics since
they map to geodesics, so by Hopf-Linow, $T_p M$ is spod.

Complete. Lemma 2 \Rightarrow covering map. \square

Variations of Energy

Idea: $p, q \in M \rightarrow$ let $\mathcal{P}(p, q) = \{\text{piecewise diff'ble paths from } p \text{ to } q\}$.

length gives a map $l: \mathcal{P}(p, q) \rightarrow \mathbb{R}_{\geq 0}$:

$$l(\gamma) = \int_a^b \langle \gamma'(t), \gamma'(t) \rangle^{1/2} dt \quad \gamma: [a, b] \rightarrow M$$

A minimizing geodesic is a global minimum for l .

More generally, a geodesic is a critical pt for l :

" $d l_\gamma: T_\gamma \mathcal{P}(p, q) \rightarrow \mathbb{R}$ " satisfies $d l_\gamma = 0$.

We'll make this precise.

First note: if γ is a length-minimizing curve from p to q ,

$$l(\gamma) \leq l(\tilde{\gamma}) \quad \forall \tilde{\gamma} \in \mathcal{P}(p, q),$$

then γ is a reparametrization of a geodesic.

Can get rid of reparam. by considering instead the energy function

$$E(\gamma) = \int_a^b \langle \gamma'(t), \gamma'(t) \rangle dt$$

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which, unlike $l(\gamma)$, changes under reparam.

Prop $\gamma: [a, b] \rightarrow M$ length-min geodesic between p and q . Then
 $E(\gamma) \leq E(\tilde{\gamma})$

for any $\tilde{\gamma}: [a, b] \rightarrow M$ between p and q , equality \Leftrightarrow
 $\tilde{\gamma}$ = length-min. geodesic.

Lemma $l(\gamma)^2 \leq (b-a)E(\gamma)$ for any path γ , equality \Rightarrow constant speed.

Pf $l(\gamma)^2 = \left(\int_a^b |\gamma'(t)| dt \right)^2 \stackrel{\text{Cauchy-Schwarz}}{\leq} \left(\int_a^b 1 dt \right) \left(\int_a^b |\gamma'(t)|^2 dt \right) = (b-a)E(\gamma),$

equality $\Leftrightarrow |\gamma'(t)|, 1$ are proportional. \square

Pf of Prop $(b-a)E(\gamma) = l(\gamma)^2 \leq l(\tilde{\gamma})^2 \leq (b-a)E(\tilde{\gamma}),$

equality $\Leftrightarrow l(\gamma) = l(\tilde{\gamma})$ and $\tilde{\gamma}$ has constant speed. \square

Next: define " $T_{\gamma} \mathcal{P}(q, j)$ " = infinitesimal deformation of a path.

Def A variation of $\gamma: [a, b] \rightarrow M$ is a smooth map

$$F: (-\epsilon, \epsilon) \times [a, b] \rightarrow M \quad \text{for some } \epsilon > 0$$

$$\text{with } F(0, t) = \gamma(t).$$

A proper variation is a variation with $F(s, a) = \gamma(a), F(s, b) = \gamma(b)$.



Variation \rightarrow Variational field $V(t) = \frac{\partial F}{\partial s}(0, t)$ vector field along γ .

Proper variation $\rightarrow V(t)$ with $V(a) = V(b) = 0$.

Prop If $V(t)$ = smooth vector field along γ + the \exists variation $F(s, t)$ (for some ϵ) s.t. $V = \dot{\gamma}(t)$ variational field. If $V(a) = V(b) = 0 \Rightarrow \exists$ proper variation.



Pf $t \in (a, b) \Rightarrow \exists \epsilon(t) \text{ s.t. } s \mapsto \exp_{\gamma(t)}(sV(t))$ is defined for $|s| < \epsilon(t)$. Compactness $\Rightarrow \exists \epsilon \text{ s.t.}$

$F(s, t) = \exp_{\gamma(t)}(sV(t))$ is defined on $(-\epsilon, \epsilon) \times [a, b]$.

- Smooth since geode is flow is smooth
- $\frac{\partial F}{\partial s}(0, t) = V(t)$ since $F(\cdot, t) = \text{geod. through } \gamma(t), \text{ initial velocity } V(t)$
- $V(a) = 0 \Rightarrow F(s, a) = \gamma(s)$. \square

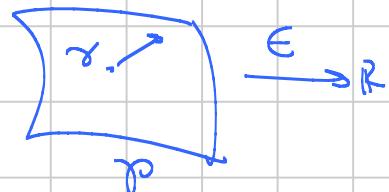
Now: Consider $E: \mathcal{P}(\rho, \gamma) \rightarrow \mathbb{R}$: want to find minima.

Want to calculate 1st and 2nd derivatives.

1st: $dE_\gamma: T_\gamma \mathcal{P}(\rho, \gamma) \rightarrow \mathbb{R}$.



$V(t)$ vecn field along γ , $V(a) = V(b) = 0$.



Prop (First variation formula)

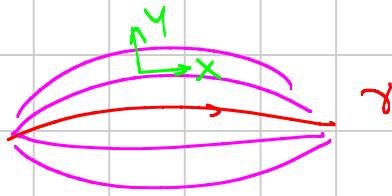
$F: (-\epsilon, \epsilon) \times [a, b] \rightarrow M$ variation of $\gamma: [a, b] \rightarrow M$, variational field $V(t)$. Then

$$\frac{1}{2} \frac{d}{ds} \Big|_{s=0} E(\gamma_s) = \left\langle V(t), \gamma'(t) \right\rangle \Big|_{t=a}^{t=b} - \int_a^b \left\langle V(t), \frac{D}{dt} \gamma'(t) \right\rangle dt$$

Notations: write $Y(s, t) = \frac{\partial F}{\partial s}$, $X(s, t) = \frac{\partial F}{\partial t}$

$$Y(0, t) = V(t), X(0, t) = \gamma'(t).$$

$$X(s, t) = \gamma'_s(t)$$



$$\text{PF } E(s) = \int_a^b \langle X, X \rangle dt$$

$$\begin{aligned} \rightarrow \frac{1}{2} E'(s) &= \frac{1}{2} \int_a^b \frac{\partial}{\partial s} \langle X, X \rangle dt \\ &= \int_a^b \left\langle \frac{\partial}{\partial s} X, X \right\rangle dt \\ &= \int_a^b \left\langle \frac{\partial}{\partial t} Y, X \right\rangle dt \\ &= \int_a^b \left(\frac{\partial}{\partial t} \langle Y, X \rangle - \langle Y, \frac{\partial}{\partial t} X \rangle \right) dt \\ &= \langle Y, X \rangle \Big|_{t=a}^{t=b} - \int_a^b \langle Y, \frac{\partial}{\partial t} X \rangle dt. \end{aligned}$$

$$\frac{DX}{ds} = \frac{D}{ds} \frac{\partial F}{\partial s} = \frac{D}{dt} \frac{\partial F}{\partial s} = \frac{DY}{dt}$$

Now plug in $s=0$. \square

Cor γ is a geodesic \Leftrightarrow \forall proper variations of γ , $E'(0)=0$.

PF. \rightarrow follows from Prop.

\Leftarrow : by Prop, \forall vector field $V(t)$ along γ with $V(a)=V(b)=0$,

$$\int_a^b \langle V(t), \nabla_{\gamma} \gamma' \rangle dt = 0$$

Choose $V(t) = f(t) \gamma'(t)$ where $f(a)=f(b)=0$, $f>0$ on (a,b) .

$$\int_a^b f(t) |\nabla_{\gamma} \gamma'|^2 dt = 0 \Rightarrow \nabla_{\gamma} \gamma' = 0. \quad \square$$

\therefore geodesics = critical pts of E . To determine if they're minimizing,
need 2nd derivative.

Prop (Second Variation Formula)

$\gamma: [a,b] \rightarrow M$ geodesic, $F: [-\epsilon, \epsilon] \times [a,b] \rightarrow M$ variation. Then

$$\frac{1}{2} E''(0) = \left\langle \frac{\partial}{\partial s} \frac{\partial F}{\partial s}(0,t), \gamma' \right\rangle \Big|_{t=a}^{t=b} + \int_a^b \left(|V'|^2 - f(V, \gamma', V, \gamma') \right) dt.$$

PF As before: $\frac{1}{2} \frac{\partial}{\partial s} \langle X, X \rangle = \langle \frac{\partial Y}{\partial t}, X \rangle$

$$\Rightarrow \frac{1}{2} \frac{\partial^2}{\partial s^2} \langle X, X \rangle = \underbrace{\left\langle \frac{\partial}{\partial s} \frac{\partial Y}{\partial t}, X \right\rangle}_{\frac{\partial}{\partial t} \frac{\partial Y}{\partial t}} + \underbrace{\left\langle \frac{\partial Y}{\partial t}, \frac{\partial X}{\partial s} \right\rangle}_{\frac{\partial Y}{\partial t}}$$

$$= \underbrace{\left\langle \frac{\partial}{\partial t} \frac{\partial Y}{\partial s}, X \right\rangle}_{\frac{\partial}{\partial t} \left\langle \frac{\partial Y}{\partial s}, X \right\rangle} - R(Y, X, Y, X) + \left| \frac{\partial Y}{\partial t} \right|^2$$

$$= 0 \text{ at } s=0 \text{ since } \gamma = \text{geodesic}$$

$$\Rightarrow \frac{1}{2} \frac{\partial^2}{\partial s^2} \langle X, X \rangle \Big|_{s=0} = \frac{\partial}{\partial t} \left\langle \frac{\partial Y}{\partial s}, \gamma' \right\rangle - R(V, \gamma', V, \gamma') + |V'|^2;$$

now integrate from $t=a$ to $t=b$. \square

Cur If $F = \text{proper variation}$ then

$$\frac{1}{2} E''(0) = \int_a^b (|V'|^2 - R(V, \gamma', V, \gamma')) dt$$

depends only on V .

Rank: $E''(0)$ = "Hessia" $d^2 E(V, V)$.

If γ is a local min for E then $E''(0) \geq 0 \forall V$.

Two applications: Myers' Thm and Synge's Thm.

Myers' (Bonnat-Myers) Thm

M complete Riem. Suppose there is $r > 0$ such that

$$\text{Ric}_p(V) \geq \frac{1}{r^2}$$

for all $p \in M$ and all $V \in T_p M$ with $|V|=1$.

Then M is compact and $\text{diam}(M) \leq \pi r = \text{diam}(S^n(r))$.

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Pf Suffices to show $\forall p, q \in M$, $\gamma = \text{min. geodesic between } p \text{ and } q \Rightarrow l(\gamma) \leq \pi r$.
 Then $\text{diam}(M) \leq \pi r \Rightarrow M \text{ bounded, complete} \Rightarrow \text{compact.}$

Suppose $l(\gamma) = l$ and assume $|\gamma'| = 1$. Choose ONB $\{e_i = \gamma'(0), e_2, \dots, e_n\}$ of $T_p M \rightarrow$ extend by parallel transport to ONB $\{e_i(t) = \gamma'(t), e_2(t), \dots, e_n(t)\}$ along γ .

Along γ , define vector field $V_i(t) = \sin\left(\frac{\pi t}{l}\right)e_i(t)$, $2 \leq i \leq n$,

$F_i = \text{proper variation of } \gamma \text{ with variational field } V_i$.

γ minimizes energy \Rightarrow by 2nd variation

$$0 \leq \frac{1}{2} E''(0) = \int_0^l (|V'_i|^2 - R(V_i, \gamma', V_i, \gamma')) dt$$

$$= \int_0^l \left(\left(\frac{\pi}{l} \cos \frac{\pi t}{l} \right)^2 - \sin^2 \left(\frac{\pi t}{l} \right) R(e_i, \gamma', e_i, \gamma') \right) dt$$

Average over i :

$$\begin{aligned} 0 &\leq \frac{\pi^2}{2r} - \int_0^l \sin^2 \left(\frac{\pi t}{l} \right) \text{Ric}_{\gamma'(t)}(e_i(t)) dt \\ &\leq \frac{\pi^2}{2r} - \int_0^l \frac{1}{r^2} \sin^2 \left(\frac{\pi t}{l} \right) dt \\ &= \frac{\pi^2}{2r} - \frac{1}{2r^2} \end{aligned}$$

$\Rightarrow l \leq \pi r$. \square

Cor M complete Riem, $\text{Ric}_p(V) \geq \frac{1}{r^2}$.

Then the universal cover of M is compact and $\pi_1(M)$ is finite.

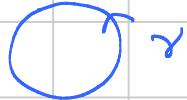
Pf Let $\tilde{M} \xrightarrow{\pi} M$ be the universal cover. Pull back metric on M to \tilde{M} ;
 π -local isometry. Then \tilde{M} is complete; apply Myers to \tilde{M}
 $\Rightarrow \tilde{M}$ cpt, # of sheets = finite. \square

Note Complete, $\text{Ric} > 0 \not\Rightarrow$ cpt, finite diameter. Ex: $\{z = x^2 + y^2\} \subset \mathbb{R}^3$.

Synge's Thm M cpt, even-dimensional, orientable,

Strictly positive sectional curvature. Then M = simply connected.

Key to Synge: closed geodesics.



A geodesic $\gamma: [0, a] \rightarrow M$ is closed/periodic if $\gamma(0) = \gamma(a)$ and $\gamma'(0) = \gamma'(a)$.

Lemma (See notes) If M isn't simply connected then there is a nontrivial minimizing closed geodesic (in any homotopy class of free loops).

Pf of Synge Let $\gamma: [0, a] \rightarrow M$ be such a geodesic. Say $\gamma(0) = \gamma(a) = p$.

Parallel transport along γ gives $P: T_p M \rightarrow T_p M$,

orientation preserving, and $P(\gamma'(0)) = \gamma'(0)$.

Let $T_p^\perp M$ = orthogonal complement to $\gamma'(0)$ in $T_p M$.

Then $P: T_p^\perp M \supseteq \{0\}$ so $P \in SO(n-1) \Rightarrow \exists v \text{ with } P(v) = v$.

Let V = parallel vector field along γ with $V(0) = V(a) = v$,

and let γ_s = corresponding variation.

$\gamma = \gamma_0$ minimizes energy $\Rightarrow E(s) = E(\gamma_s)$ has local min at $s=0$.

But

$$\begin{aligned} \frac{1}{2} E''(0) &= \left(\frac{D}{ds} \frac{\partial F}{\partial s}(0, t), \gamma' \right) \Big|_{t=0}^{t=a} + \int_0^a (|V'|^2 - R(V, \gamma', V, \gamma')) dt \\ &\quad \xrightarrow{\text{Since } \gamma_s \text{ periodic}} \\ &= - \int_0^a R(V, \gamma', V, \gamma') dt \\ &\quad \xrightarrow{\gg 0} \\ &< 0. \end{aligned}$$

$\Rightarrow \square$

4/23 \square

Synge's Thm M cpt, even-dimensional, orientable,

Strictly positive sectional curvature. Then $M = \text{Simply connected.}$

Remarks 1. Statement in odd dim: M cpt, odd-dim, $K > 0$. Then M is orientable. (see book)

2. Can't remove even-dim or orientable assumption:
 \mathbb{RP}^n has $K > 0$ with metric induced from S^n .
Can't weaken to $K \geq 0$: $T^n = \mathbb{R}^n / \mathbb{Z}^n$.

Key to Synge: closed geodesics.



A geodesic $\gamma: [0, a] \rightarrow M$ is closed if $\gamma(0) = \gamma(a)$ and $\gamma'(0) = \gamma'(a)$:
think of this as a smooth map $S^1 \rightarrow M$.

∅

Def A homotopy class of free loops in M is a map $S^1 \rightarrow M$ up to homotopy.

Prop M compact. In any homotopy class of free loops in M , there is a closed geodesic.

Pf Cpt $\Rightarrow \exists \epsilon > 0$ with $B_\epsilon(p)$ normal $\forall p \in M$.

Consider $l = \inf \{l(\gamma) \mid \gamma \in \text{htpy class}\}$.

If $\exists \gamma$ with $l(\gamma) = l$ then γ locally minimizes length \Rightarrow geodesic.

Otherwise, assume $\exists \gamma_i$ with $l(\gamma_i) \rightarrow l$. Reparametrize so

$\gamma_i: [0, 1] \rightarrow M$, $|\gamma'_i| = l(\gamma_i)$.

Choose $0 = t_0 < t_1 < \dots < t_N = 1$ w.k.t $t_{k+1} - t_k < \frac{\epsilon}{\max(\gamma_i')}$.

Then $\gamma_i(t_k), \gamma_i(t_{k+1})$ lie in a normal ϵ -ball

so we can replace γ_i by piecewise smooth curve s.t.
 $\gamma_i|_{[t_k, t_{k+1}]}$ is a geodesic.

By passing to a subsequence, can assume $\forall k, \gamma_i(t_k) \rightarrow p_k$
 $\gamma_i'(t_k) \rightarrow v_k$

Now define $\gamma: [0, 1] \rightarrow M$ by $\gamma|_{[t_k, t_{k+1}]} =$ geodesic starting at p_k with deriv = v_k .

Then $\gamma: [t_k, t_{k+1}] \rightarrow \gamma|_{[,]}$ so $l(\gamma) = \lim l(\gamma_i) = l$. \square

Pf of Uniq Suppose M not simply connected. Then \exists closed

geodesic $\gamma: [0, a] \rightarrow M$ in a nontrivial homotopy class of minimum length.

Say $\gamma(0) = \gamma(a) = p$.

Parallel transport along γ gives $P: T_p M \rightarrow T_p M$,

orientation preserving, and $P(\gamma'(0)) = \gamma'(0)$.

Let $T_p^\perp M =$ orthogonal complement to $\gamma'(0)$ in $T_p M$.

Then $P: T_p^\perp M \supseteq \Rightarrow P \in SO(n-1) \Rightarrow \exists v$ with $P(v) = v$.

let $V =$ parallel vector field along γ with $V(0) = V(a) = v$,

and let $\gamma_s =$ corresponding variation.

$\gamma = \gamma_0$ minimizes energy $\Rightarrow E(s) = E(\gamma_s)$ has local min at $s=0$.

But

$$\begin{aligned} \frac{1}{2} E''(0) &= \left(\frac{\partial^2 E}{\partial s^2}(0, t), \gamma' \right) \Big|_{t=0} + \int_0^a (|V'|^2 - R(V, \gamma', V, \gamma')) dt \\ &\stackrel{\textcircled{O} \text{ since } \gamma_s \text{ periodic}}{=} - \int_0^a R(V, \gamma', V, \gamma') dt \\ &< 0. \end{aligned}$$

$\Rightarrow \square$

Constant Sectional Curvature

Three manifolds of constant sectional curvature

- \mathbb{R}^n , flat : $K = 0$
 - S^n , round : $K = 1$
 - H^n , hyperbolic : $K = -1$ (see HW)
- } all complete,
simply connected

If (M, g) has constant K , we can rescale to get $K \in \{1, 0, -1\}$:

if $\lambda > 0$ then $\tilde{g} = \lambda g$ is a metric with $\tilde{\nabla} = \nabla$, $\tilde{R} = \lambda R$, $\tilde{K} = \lambda^4 K$.

$(0, 4)$ -term

Can get more by quotienting by a group of isometries.

Thm M^n Complete Riem mfd, Constant sectional curvature

$K \in \{1, 0, -1\}$. Then its universal cover is:

1. \mathbb{R}^n , flat if $K = 0$
2. S^n , round if $K = 1$
3. H^n , hyperbolic if $K = -1$,

i.e. any complete mfd with constant K is a quotient of one of them by isometries.

Pf Idea: $p \in M \rightsquigarrow \exp_p: T_p M \rightarrow M$ is a local isometry if

$K=0$, almost a local isometry if $K=\pm 1$. Choose $v \in T_p M$

unit vector.



Look at $(d\exp_p)_{tv} : T_{tv}(T_p M) \rightarrow T_{\exp_p(tv)} M$.

Note this maps

$$\begin{aligned} T_p M & \\ \sqrt{ } & \longmapsto \gamma'(t) \quad (\text{these have the same length}) \\ \sqrt{ }^\perp & \longmapsto (\gamma'(t))^\perp \quad \text{by Gauss Lemma.} \end{aligned}$$

So it suffices to see what this does on $\sqrt{ }^\perp \subset T_p M$.

Choose $u \in T_p M$ with $u \perp \sqrt{ }$.

$\rightsquigarrow U(t)$ = parallel vector field along γ with $U(0) = u$;

$J(t)$ = Jacobi field with $J(0) = 0$, $J'(0) = u$:

$$J(t) = (d\exp_p)_{tv}(tu).$$



How are these related? Jacobi equation $J'' = -R(\gamma', J)\gamma'$,

constant sectional curvature $\Rightarrow R(XY\gamma Z) = K(\langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle)$

$$\Rightarrow R(\gamma', J)\gamma' = K(\langle \gamma', \gamma' \rangle J - \langle J, \gamma' \rangle \gamma') = KJ$$

$$\Rightarrow \boxed{J'' = -KJ}.$$

Case 1. $K=0$: already done in HW.

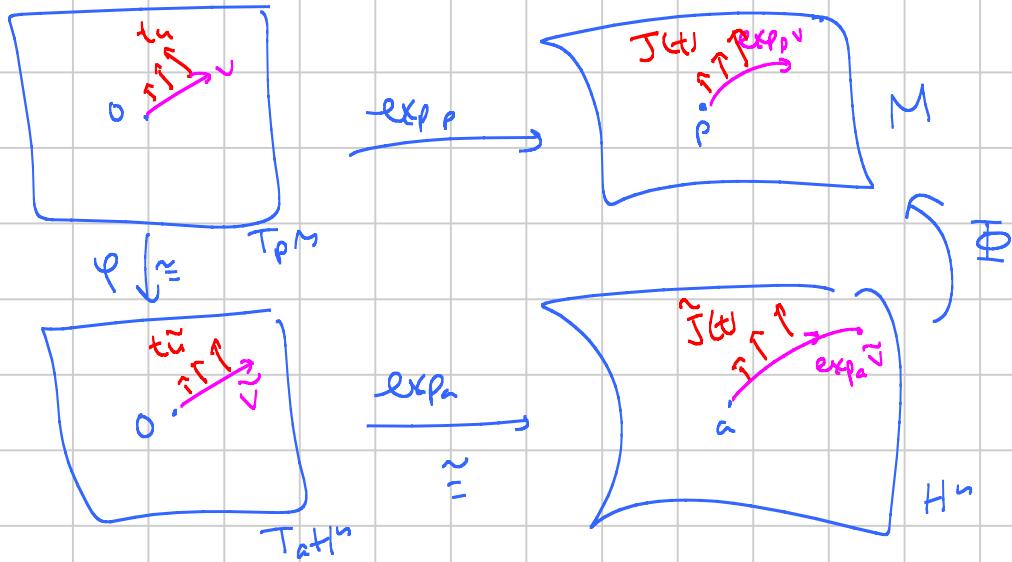
$$J'' = 0 \Rightarrow J(t) = tU(t)$$

$$\Rightarrow (d\exp_p)_{tv}(u) = U(t)$$

And parallel transport preserves inner product, so $(d\exp_p)_{tv}$ is an isometry on $\sqrt{ }^\perp \Rightarrow$ on all of $T_{tv}(T_p M)$.

Thus \exp_p is a local isometry $T_p M \xrightarrow[R^+]{\sim} M$, so by the proof of Hadamard, it's a covering map.

Case 2 $\lambda = -1$. $J' = +J \Rightarrow J(t) = (\sinh t) U(t)$.



Choose $a \in H^n$ and pick a linear isometry $\varphi: T_p M \rightarrow T_a H^n$.

Claim: $\Phi := \exp_p^{-1} \circ (\exp_a)^{-1} \circ \varphi$ is a local isometry
 $H^n \rightarrow M \quad \Rightarrow$ Covering map.

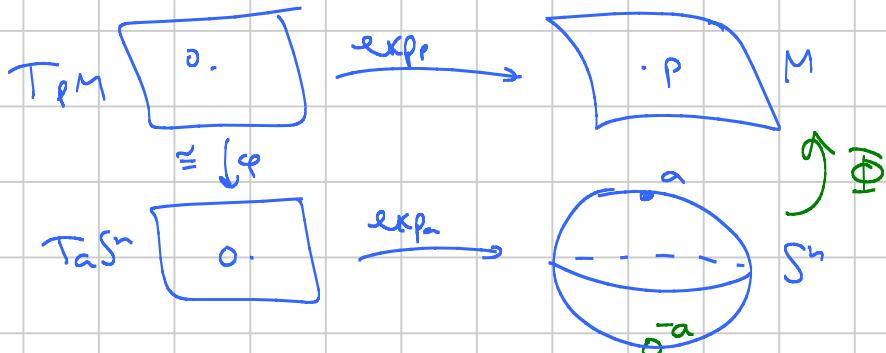
PF Let $\tilde{v} \in T_a H^n$ be any unit vector, $v = \varphi^{-1}(\tilde{v})$ = unit vector in $T_p M$.
Want to show: isometry at $\exp_a(t\tilde{v})$. Suffices to show on \perp .

Choose $u \perp v$, $\tilde{u} = \varphi(u) \perp \tilde{v} \Rightarrow$ Jacobi fields $J(t) = d(\exp_p)_{tu}(t u)$,
 $\tilde{J}(t) = d(\exp_a)_{t\tilde{v}}(t\tilde{v})$. Note $J(t) = \Phi_* \tilde{J}(t)$.

Now if we have $u_1, u_2 \perp v \rightsquigarrow J_1(t), J_2(t)$ on M , $\tilde{J}_1(t), \tilde{J}_2(t)$ on H^n ,
then suffices to show $\langle J_1(t), J_2(t) \rangle = \langle \tilde{J}_1(t), \tilde{J}_2(t) \rangle$.

$J_i(t) = (\sinh t) U_i(t) \Rightarrow \langle J_1(t), J_2(t) \rangle = (\sinh t)^2 \langle U_1(t), U_2(t) \rangle = (\sinh t)^2 \langle u_1, u_2 \rangle$
and similarly $\langle \tilde{J}_1(t), \tilde{J}_2(t) \rangle = (\sinh t)^2 \langle \tilde{u}_1, \tilde{u}_2 \rangle$ $\overbrace{\qquad}$
as desired.

Case 3 $K=1$. $J'=-J \Rightarrow J(t)=(\sin t)U(t)$.

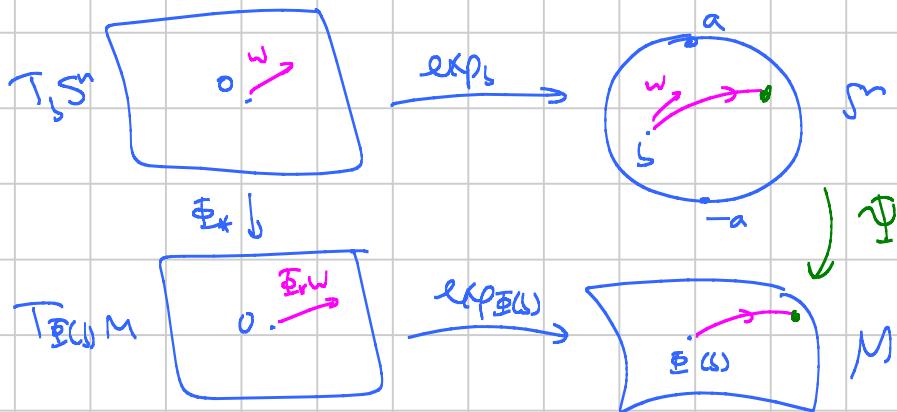


\exp_a differs from $B_\pi(0)$ to $S^n - \{-a\}$.

Same argument as before: $\Phi := \exp_p \circ \varphi^{-1} \circ (\exp_a)^{-1}: S^n - \{-a\} \rightarrow M$ is a local isometry.

Now choose $b \neq \pm a$ in S^n . Define

$$\Psi := \exp_{\Phi(b)} \circ \Phi_* \circ (\exp_b)^{-1}: S^n - \{-b\} \rightarrow M.$$



Claim: $\Phi = \Psi$. Why? A geodesic from b to c with tangent vector w maps under both Φ and Ψ to a geodesic from $\Phi(b)$ with tangent vector $\Phi_*(w)$, so Φ, Ψ map c to the same thing.

So Φ, Ψ together give $\Phi: S^n \rightarrow M$ local isometry
 \rightarrow covering map, as before. \square

2
 4/25 and end of material for 2018 class

Cut Locus

M complete $\Rightarrow \exp_p: T_p M \rightarrow M$ surjective. Can we always model M by part of $T_p M$? e.g. for S^n , $\exp_p: B_{\pi}(0) \rightarrow S^n - \{-p\}$ diffeo.

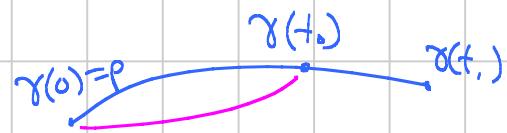
For a geodesic $\gamma(t)$, $0 \leq t < \infty$, starting at p , we know that $\gamma|_{[0,t]}$ is a minimizing geodesic for small t . Define

$$I = \{t_0 \geq 0 \mid \gamma|_{[0,t_0]} \text{ is a minimizing geodesic}\} \\ i.e. t_0 = d(p, \gamma(t_0))$$

- I is closed

- if $t_0 \notin I$ and $t_1 > t_0$ then $t_1 \notin I$:

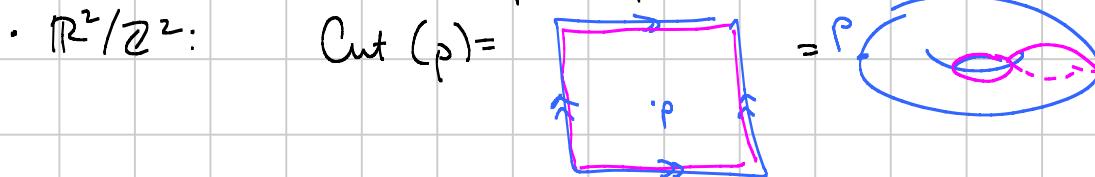
So either $I = [0, T]$ for some $T > 0$ or $I = [0, \infty)$.



Def The cut point of p along γ is $\gamma(T)$ (if $I = [0, \infty)$, no cut point).

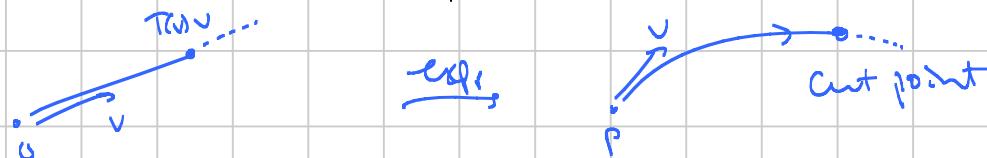
The Cut Locus of p is $\text{Cut}(p) = \{\text{cut points over all } \gamma\}$.

Ex: round sphere: $\text{Cut}(p) = \{-p\}$



- Complete, Simply Connected, $K \leq 0 \rightarrow \text{Cut}(p) = \emptyset$. (Hadamard)

Now let the geodesic vary: $v \in T_p M, |v|=1 \rightarrow \exp_p(tv)$



Define $T(v) = \begin{cases} T & \text{if } I = [0, T] \\ \infty & \text{if } I = [0, \infty) \end{cases}$.

The preimage of the geodesic up to the cut point is the ray to $T(v)$ v.

Write $U(p) = \cup(\text{these open rays})$ i.e.

$$\begin{aligned} U(p) &= \{tv \mid v \in S^{n-1}, 0 \leq t < T(v)\} \\ &= \{t_v \mid N=1, \exp_p(tv) \text{ is minimizing past } t\}. \end{aligned}$$



- $U(p)$ is star-shaped
- Can show $T: S^{n-1} \rightarrow \mathbb{R}^+ \cup \{\infty\}$ is continuous $\Rightarrow U(p) \cong D^n$ (in fact, diffeo)
- $\text{Cut}(p) = \exp_p(\partial U(p))$

homeo

Prop M (complete) is the disjoint union $\exp(U(p)) \sqcup \text{Cut}(p)$.

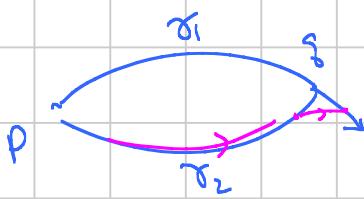
Pf $g \in M \Rightarrow \exists$ min geodesic from p to g (Hopf-Rinow).

Either this stops being minimizing past $g \Rightarrow g \in \text{Cut}(p)$
or not $\Rightarrow g \in \exp_p(U(p))$.

If $g \in \exp_p(U(p)) \cap \text{Cut}(p)$ then $\exists 2$ minimizing geodesics between p and g one minimizing past g , one not. Prop now follows from
the next result. \square

Lemma If \exists 2 minimizing geodesics between p and q , then neither minimizes past q .

Pf.



$l(\gamma_1) = l(\gamma_2)$. Extend γ_2 past q .

A shorter path is $\gamma_2 \cup$ cutoff. \square

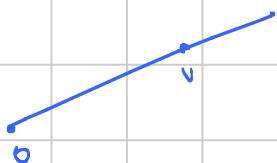
Note: this argument actually shows

\exp_p is an injection
+ a bijection

$U(p) \rightarrow M$
 $U(p) \rightarrow M \setminus \text{Cut}(p)$.

Prop $\exp_p: U(p) \rightarrow M \setminus \text{Cut}(p)$ is a diffeomorphism.

Pf



$v \in U(p) \subset T_p M \rightarrow \gamma(t) = \exp_p(tv)$ is minimizing

path $t=1$. Then γ has no conjugate points to p

between p and $\exp_p((1+\epsilon)v)$ (from HW) so $\exp_p v$ isn't

conjugate to $p \rightarrow \exp_p$ is a local diffeomorphism at v .

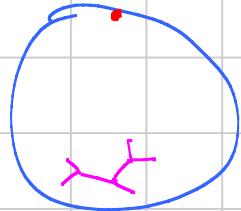
$\Rightarrow \exp_p$ is a local diffeo on $U(p)$

\Rightarrow diffeo since bijective. \square

All the interesting topology in M is in $\text{Cut}(p)$:

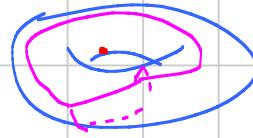
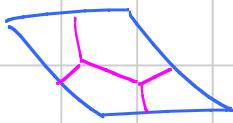
$$\begin{aligned} \exp_p: T^* \cong U(p) &\xrightarrow{\sim} M \setminus \text{Cut}(p) \\ \partial U(p) &\longrightarrow \text{Cut}(p). \end{aligned}$$

Weird examples: S^n :



Note M_p deformation retracts onto $\text{Cut}(p)$.

$$T^2 = \mathbb{R}^2 / \mathbb{Z}^2 = \mathbb{C}^2 / \langle 1, e^{2\pi i/3} \rangle :$$



Prop (see do Carmo Ch 13 Prop 2.2)

$g = \text{Cut point of } p \text{ along } \gamma \Leftrightarrow$ first point along γ where either

- Conjugate to p
- \exists two minimizing geodesics between p and g .

$g \in M \setminus \text{Cut}(p) \Rightarrow \exists!$ min geodesic from p to g

$\leftarrow (19)$ $\Rightarrow \exp_p: B_r(o) \rightarrow B_r(p)$ is injective $\Leftrightarrow r \leq d(p, \text{Cut}(p))$.

Def The injectivity radius $i(M) := \inf_{p \in M} d(p, \text{Cut}(p))$.

(if $r \leq i(M)$ then $\exp_p: B_r(o) \rightarrow M$ is injective $\forall p$).

Intuitively: if $K = \text{small}$ then $i = \text{large}$. Sample them.

Thm (Ch 13 Prop 2.13) If $0 < a \leq K \leq K_{\max}$ for some a, K_{\max} ,
then either \exists closed geodesic γ with $i(M) = \frac{1}{2}l(\gamma)$ or $i(M) \geq \frac{\pi}{\sqrt{K_{\max}}}$.
(Note by Myers that M is cpt.)

Thm (Klingenberg) M simply connected, cpt, $\dim \geq 3$,
 $\frac{K_0}{4} < K \leq K_0$ for constant $K_0 > 0$.

Then $i(M) \geq \frac{\pi}{\sqrt{K_0}}$. (idea: rule out short geodesics)

Sphere Theorem M simply connected, cpt,
 $0 < \frac{K_0}{4} < K \leq K_0$.

Then M is homeomorphic to S^n .

Submanifolds

$M^n \subset \bar{M}^{n+r}$ submfld, Riem metric on $\bar{M} \rightarrow$ Riem. metric on M .

Levi-Civita connections $\bar{\nabla}, \bar{\nabla}$.

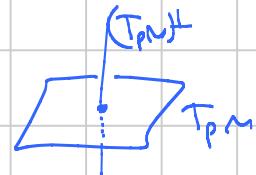
Recall: $X, Y \in \text{Vect}(M)$ extending to $\bar{X}, \bar{Y} \in \text{Vect}(\bar{M})$

$$\Rightarrow \bar{\nabla}_X Y = (\bar{\nabla}_{\bar{X}} \bar{Y})^\top$$

independent of the extensions \bar{X}, \bar{Y} (HU).

$$\text{where } p \in M \rightsquigarrow T_p \bar{M} = T_p M \oplus (T_p M)^\perp$$

$$v \mapsto v^T \oplus v^N$$



Def $X, Y \in \text{Vect}(M)$

$$\rightsquigarrow B(X, Y) := (\bar{\nabla}_X \bar{Y})^N = \bar{\nabla}_{\bar{X}} \bar{Y} - \bar{\nabla}_Y \bar{X} \quad \text{vector field along } M.$$

"Vector valued second fundamental form" (in $(T_p M)^\perp$).

Facts.

1. Independence of extension \bar{X}, \bar{Y} .

as in HW: if \bar{X}, \bar{X}' are extensions of X , $\bar{\nabla}_{\bar{X}} \bar{Y} - \bar{\nabla}_{\bar{X}'} \bar{Y} = \bar{\nabla}_{\bar{X}-\bar{X}'} \bar{Y} = 0$
since $\bar{X}-\bar{X}'=0$ on M

\bar{Y}, \bar{Y}' extensions of $Y \Rightarrow \bar{\nabla}_{\bar{X}} \bar{Y} - \bar{\nabla}_{\bar{X}} \bar{Y}' = \bar{\nabla}_{\bar{X}} (\bar{Y} - \bar{Y}') = 0$ since $\bar{Y} - \bar{Y}' = 0$ on M
 \bar{X} tangent to M .

2. tensor: clearly tensorial in X .

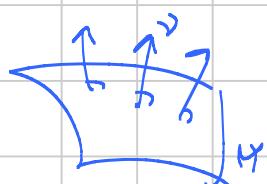
$$\begin{aligned} B(X, fY) &= \bar{\nabla}_{\bar{X}}(f\bar{Y}) - \bar{\nabla}_X(fY) = f B(X, Y) + (\bar{X}f)\bar{Y} - (Xf)Y \\ &= f B(X, Y) \end{aligned}$$

3. Symmetric:

$$B(X, Y) - B(Y, X) = [\bar{X}, \bar{Y}] - [X, Y] = 0 \text{ on } M.$$

Choose a unit normal vector field v along M .

(if M, \bar{M} orientable, then v is well-defined globally up to ± 1).



Def The Second fundamental form is the symmetric bilinear form

$\mathbb{II}: T_p M \otimes T_p M \rightarrow \mathbb{R}$ given by

$$B(X, Y) = \mathbb{II}(X, Y) v.$$

(Note: 1st fundamental form is just the metric)

Def σ = 2-plane in $T_p M$ generated by X, Y .

The Gaussian Curvature $G(\sigma)$ is

$$G(\sigma) := \frac{\mathbb{II}(X, X)\mathbb{II}(Y, Y) - \mathbb{II}(X, Y)^2}{\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2}.$$

This is independent of choice of X, Y (as in sectional curv.).

Def The shape operator $S: T_p M \rightarrow T_p M$ is defined by

$$\langle S(X), Y \rangle = \mathbb{II}(X, Y).$$

Alternate def for S : note

$$\begin{aligned} \mathbb{II}(X, Y) &= \langle B(X, Y), v \rangle = \langle \bar{\nabla}_X Y - \bar{\nabla}_Y X, v \rangle = \langle \bar{\nabla}_X Y, v \rangle \\ &= \bar{X} \langle \bar{Y}, v \rangle - \langle \bar{Y}, \bar{\nabla}_X v \rangle = -\langle \bar{Y}, \bar{\nabla}_X v \rangle \\ \Rightarrow S(X) &= -(\bar{\nabla}_X v)^T. \end{aligned}$$

Eigenvalues of S are principal Curvatures.

For surfaces in \mathbb{R}^3 , $G(\sigma) = \lambda_1 \lambda_2$ product of eigenvalues.

Then (Gaus) $X, Y, U, V \in T_p M$. The

$$R(X, Y, U, V) = \bar{R}(X, Y, U, V) + \mathbb{II}(X, U)\mathbb{II}(Y, V) - \mathbb{II}(X, V)\mathbb{II}(Y, U)$$

Sectional Gaussian curvature of plane σ \rightarrow

$$K(X, Y) = \bar{K}(X, Y) + G(X, Y).$$

PF Extend X, Y, U, V at p to vector fields $\bar{X}, \bar{Y}, \bar{U}, \bar{V}$ on \bar{M} ,

$\bar{X}, \bar{Y}, \bar{U}, \bar{V}$ on \bar{M} .

$$\begin{aligned}\bar{\nabla}_{\bar{X}} \bar{Y} &= \bar{\nabla}_X Y + \mathbb{I}(X, Y) v \quad \text{on } M \\ \Rightarrow \bar{\nabla}_{\bar{X}} \bar{\nabla}_{\bar{Y}} \bar{U} &= \bar{\nabla}_{\bar{X}} (\bar{\nabla}_Y U + \mathbb{I}(Y, U) v) \\ &= \bar{\nabla}_X \bar{\nabla}_Y U + \mathbb{I}(X, \bar{\nabla}_Y U) v + \mathbb{I}(Y, U) \bar{\nabla}_{\bar{X}} v + (\bar{X} \mathbb{I}(Y, U)) v \\ \Rightarrow \langle \bar{\nabla}_{\bar{X}} \bar{\nabla}_{\bar{Y}} \bar{U}, \bar{V} \rangle &= \langle \bar{\nabla}_X \bar{\nabla}_Y U, V \rangle + \mathbb{I}(Y, U) \bar{\nabla}_{\bar{X}} v, \bar{V} \rangle \\ &= \langle \bar{\nabla}_X \bar{\nabla}_Y U, V \rangle - \mathbb{I}(Y, U) \mathbb{I}(Y, V). \quad \square\end{aligned}$$

Important special case:

Then If $\bar{M} = (\mathbb{R}^{n+1}, \text{flat})$ then $K(X, Y) = G(X, Y)$.

Rank For $n=3$ this is Gauss's Theorema Egregium.

in his language: the extrinsic quantity $G(X, Y)$ (depends on the isometric embedding of M in \mathbb{R}^3) is actually an intrinsic quantity.

Ex $S^n(r) \subset \mathbb{R}^{n+1}$.

$$x \in S^n(r) \Rightarrow y(x) = \frac{x}{r} = \frac{1}{r}(x^i \partial_i). \quad \text{For } Y \in T_x(S^n(r)),$$

$$\bar{\nabla}_Y v = Yv = \frac{1}{r}(Y(x^i) \partial_i) = \underbrace{\frac{Y}{r}}_{Y_r=0}$$

$$\text{so } S(Y) = -\frac{Y}{r} \Rightarrow \mathbb{I}(X, Y) = -\frac{1}{r} \langle X, Y \rangle$$

$$\text{and } G(X, Y) = \frac{1}{r^2} \Rightarrow K = \frac{1}{r^2}.$$

Totally geodesic submanifolds

Def $M \subset \bar{M}$ is totally geodesic if every geodesic in M is also a geodesic in \bar{M} .

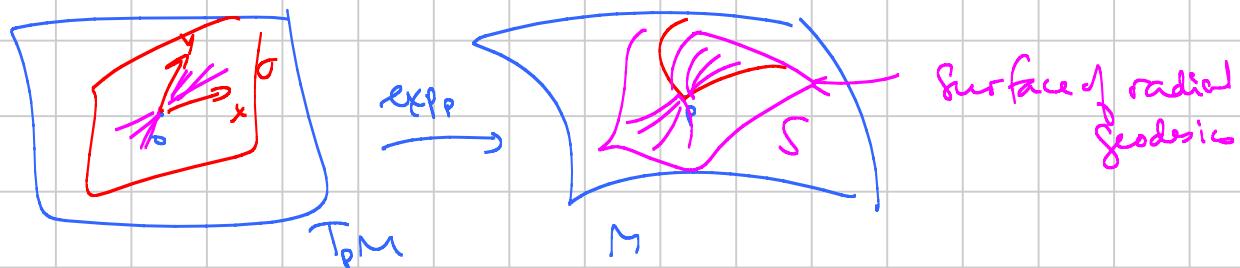
Prop M is totally geodesic $\Leftrightarrow B \equiv 0$ (vector valued 2nd fund. form).

Pf let γ be a path in M , $\gamma(0) = p$, $\gamma'(0) = x$. Let N be a normal vector field to M , X = extension of x .

$$\begin{aligned} \langle B(x, x), N \rangle &= \langle \bar{\nabla}_x X - \nabla_x X, N \rangle \\ &= \langle \bar{\nabla}_x X, N \rangle \end{aligned}$$

so $B(x, x) = 0 \Leftrightarrow \bar{\nabla}_x X$ has no normal component
 \Leftrightarrow geodesics in M (satisfying $\nabla_x X = 0$)
are also geodesic in \bar{M} (satisfying $\bar{\nabla}_X X = 0$). \square

Riemann's original definition of sectional curvature:



S is geodesic at p (geodesic version of totally geodesic)

\Rightarrow by above Prop, $B(p) = 0$

\Rightarrow Give S the metric induced by M ; then $R = \bar{R}$

and $K(S) = K(\sigma)$

(unique) sectional curvature of S \curvearrowleft usual sectional curvature