

Math 621 HW 9 — Outline of Solutions

Note Title

4/16/2018

1. (a) Write $K: M \rightarrow \mathbb{R}$ with $K(\sigma) = K(p)$ for $\sigma \in T_p M$. As we showed in class, for $X, Y, Z, W \in T_p M$,

$$(*) \quad R(X, Y, Z, W) = K(p) (\langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle).$$

(In class we assumed $K(p) \equiv K_0$, indep. of p , but the same proof applies here.)

It follows that

$$\begin{aligned} (\nabla_X R)(Y, Z, W, T) &= X R(Y, Z, W, T) - R(\nabla_X Y, Z, W, T) - R(Y, \nabla_X Z, W, T) \\ &\quad - R(Y, Z, \nabla_X W, T) - R(Y, Z, W, \nabla_X T) \\ &= (XK)(p) R(Y, Z, W, T) + K(p) [X(\langle Y, W \rangle \langle Z, T \rangle - \langle Y, T \rangle \langle W, Z \rangle) \\ &\quad - R(\nabla_X Y, Z, W, T) - R(Y, \nabla_X Z, W, T) - R(Y, Z, \nabla_X W, T) \\ &\quad - R(Y, Z, W, \nabla_X T)] \end{aligned}$$

this is 0 by (a)

and second Bianchi gives

$$\begin{aligned} 0 &= (\nabla_X R)(Y, Z, W, T) + (\nabla_Y R)(Z, X, W, T) + (\nabla_Z R)(X, Y, W, T) \\ &= (XK)(\langle Y, W \rangle \langle Z, T \rangle - \langle Y, T \rangle \langle Z, W \rangle) + (YK)(\langle Z, W \rangle \langle X, T \rangle - \langle Z, T \rangle \langle X, W \rangle) \\ &\quad + (ZK)(\langle X, W \rangle \langle Y, T \rangle - \langle X, T \rangle \langle Y, W \rangle). \end{aligned}$$

Now fix $X(p), Y(p)$ and choose $T(p) = Z(p)$ to have length 1 and be orthogonal to $X(p)$ and $Y(p)$ (this can be done if $\dim M \geq 3$). Then $(**)$ becomes

$$0 = (XK) \langle Y, W \rangle - (YK) \langle X, W \rangle.$$

This can only happen for all W if $(XK)Y = (YK)X$, and this can happen for all X, Y only if $(XK)(p) = 0 \quad \forall X \in T_p M$.

Finally, since this is true for all p , $K = \text{constant}$.

1. (b) Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for $T_p M$. Then
 Cont'd. $\text{Ric}(e_1, e_1) = \frac{1}{2} (R(e_1, e_2, e_1, e_2) + R(e_1, e_3, e_1, e_3))$

and similarly for $\text{Ric}(e_2, e_2)$ and $\text{Ric}(e_3, e_3)$.

If $\sigma =$ plane generated by e_1 and e_2 , then

$$(***) \quad K(\sigma) = R(e_1, e_2, e_1, e_2) = \text{Ric}(e_1, e_1) + \text{Ric}(e_2, e_2) - \text{Ric}(e_3, e_3).$$

Given σ , we can choose $\{e_1, e_2\} = \text{ONB}$ for σ and complete to

$\{e_1, e_2, e_3\} = \text{ONB}$ for $T_p M$ then by (***) $K(\sigma)$ is determined by Ric .

2. Suppose $v \in B_\epsilon(0) \subset T_p M$. We want to show, for $u_1, u_2 \in T_v(T_p M) = T_p M$,
 $\langle u_1, u_2 \rangle = \langle (d\exp_p)_v u_1, (d\exp_p)_v u_2 \rangle_{\exp_p v}$.

Let J be the Jacobi field along $\gamma(t) = \exp_p(tv)$ with $J(0) = 0, J'(0) = u_1$.

The Jacobi equation is $J'' = 0$ so if $U_1 =$ parallel vector field along γ with $U_1(0) = u_1$, then $J(t) = tU_1(t)$. On the other

hand, $J(t) = (d\exp_p)_{tv}(tu_1)$, so

$$U_1(1) = (d\exp_p)_v(u_1).$$

Similarly, if $U_2 =$ parallel vector field with $U_2(0) = u_2$, then

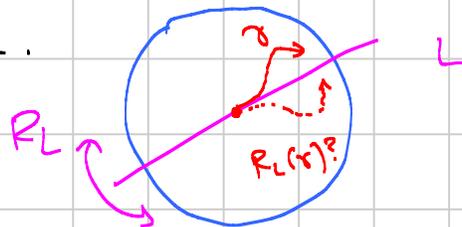
$$U_2(1) = (d\exp_p)_v(u_2).$$

Thus

$$\langle (d\exp_p)_v(u_1), (d\exp_p)_v(u_2) \rangle = \langle U_1(1), U_2(1) \rangle \stackrel{\substack{\text{parallel transport} \\ \text{preserves } \langle, \rangle}}{=} \langle u_1, u_2 \rangle.$$

3. (a) Routine computation.

(b) If γ is a nonconstant geodesic in D^2 with $\gamma(0) = (0,0)$, then since reflection R_L in the line L through $(0,0)$ in the direction of $\gamma'(0)$ is an isometry, and R_L preserves the initial conditions of γ , γ must be invariant under the action of R_L ; so γ has to lie in the line L .



Now we use Jacobi fields to calculate sectional curvature.

Let $v \in T_0 D^2$ be a unit vector (in the hyperbolic metric): $v_1^2 + v_2^2 = \frac{1}{4}$ where $v = (v_1, v_2)$. Write $\gamma(t) = f(t)v$ for the geodesic with $\gamma(0) = 0$, $\gamma'(0) = v$; assume $f(t)$ increasing. Then γ has constant speed 1

$$\Rightarrow 1 = |\gamma'(t)| = |f'(t)v|_{\gamma(t)} = \left| (f'(t)v_1, f'(t)v_2) \right|_{(f(t)v_1, f(t)v_2)} = \frac{f'(t)}{1 - \left(\frac{f(t)}{2}\right)^2}$$

$$\Rightarrow (\text{elementary differential equations}) \quad \underline{f(t) = 2 \tanh\left(\frac{t}{2}\right)}$$

Now let $v = (\frac{1}{2}, 0)$, $w = (0, \frac{1}{2}) \in T_0 D^2$. The path $s \mapsto (\cos s)v + (\sin s)w =: V(s)$ consists of unit vectors in $T_0 D^2$, so the geodesic γ_s with $\gamma_s(0) = 0$, $\gamma_s'(0) = V(s)$ is given by

$$\gamma_s(t) = 2 \left(\tanh\left(\frac{t}{2}\right) \right) (\cos s)v + (\sin s)w$$

and the corresponding Jacobi field is

$$J(t) = \frac{\partial}{\partial s} \Big|_{s=0} \gamma_s(t) = 2 \left(\tanh\left(\frac{t}{2}\right) \right) w$$

(this is a vector at the point $2 \left(\tanh\left(\frac{t}{2}\right) \right) v$)

$$\Rightarrow |J(t)|^2 = \frac{4 \left(\tanh\left(\frac{t}{2}\right) \right)^2}{\left(1 - \left(\tanh\left(\frac{t}{2}\right) \right)^2 \right)^2} = t^2 + \frac{1}{3}t^4 + \dots$$

$$\Rightarrow R(v, w, v, w) = -1$$

$$\Rightarrow \text{Sectional curvature} = -1.$$

\uparrow
 v, w are orthonormal at 0

4. (a) For $x \in \mathbb{H}^n$, $T_x \mathbb{H}^n \cong \{y \in \mathbb{R}^{n+1} \mid \langle x, y \rangle = 0\}$.

[Proof: if $x(t)$ is a path in \mathbb{H}^n with $x(0) = x$ then $0 = \frac{d}{dt} \Big|_{t=0} \langle x(t), x(t) \rangle = 2 \langle x, x'(0) \rangle$ so $T_x \mathbb{H}^n \subset \{\langle x, y \rangle = 0\}$, and both of these are n -dimensional vector spaces.]

Since $\langle x, x \rangle = -1$ and $T_x \mathbb{H}^n$ is the orthogonal complement of x in \mathbb{H}^n with respect to $\langle \cdot, \cdot \rangle$ (which has signature $(n, 1)$), Sylvester implies that $\langle \cdot, \cdot \rangle$ must be positive definite on $T_x \mathbb{H}^n$.

(b) Define $f: \mathbb{H}^n \rightarrow \mathbb{R}^{n+1}$ by

$$f(x) = \zeta - \frac{2(x-\zeta)}{\langle x-\zeta, x-\zeta \rangle}$$

where $\zeta = (-1, 0, \dots, 0)$. We claim that f maps \mathbb{H}^n diffeomorphically to $D^n = \{(0, x_1, \dots, x_n) \mid x_1^2 + \dots + x_n^2 < 1\}$. Note $f = f^{-1}$.

In coordinates we have

$$f(x_0, x_1, \dots, x_n) = (y_1, \dots, y_n) \quad (\text{drop the initial } 0)$$

where

$$y_1 = \frac{x_1}{x_0+1}, \dots, y_n = \frac{x_n}{x_0+1}$$

$$\Rightarrow y_1^2 + \dots + y_n^2 = \frac{1-x_0^2}{(x_0+1)^2} = \frac{1-x_0}{1+x_0} < 1$$

with inverse $x_0 = \frac{1-|y|^2}{1+|y|^2}$, $x_1 = \frac{2y_1}{1+|y|^2}$, \dots , $x_n = \frac{2y_n}{1+|y|^2}$ where $|y|^2 = y_1^2 + \dots + y_n^2$

$$\Rightarrow -x_0^2 + x_1^2 + \dots + x_n^2 = -1$$

$$\text{so } f(\mathbb{H}^n) \subset D^n, f^{-1}(D^n) \subset \mathbb{H}^n.$$

Then compute directly that f maps $\langle \cdot, \cdot \rangle$ to the hyperbolic metric on D^n .

4. (c) Given $x \in H^n \subset \mathbb{R}^{n+1}$, let $x(t)$ be a path in H^n with
 Cont'd. $x(0) = p$, $x(1) = x$ where $p = (1, 0, \dots, 0)$. Note $\langle \cdot, \cdot \rangle$ is positive
 definite on the orthogonal complement $(x(t))^\perp \subset \mathbb{R}^{n+1}$; so we can
 choose a continuously varying orthonormal basis
 $e_1(t), \dots, e_n(t)$ of $(x(t))^\perp$ with $e_i(0), \dots, e_n(0) =$ standard basis
 of $(x(0))^\perp = \{(0, x_1, \dots, x_n)\}$.

Then the map $\varphi_t: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ defined by

$$\varphi_t(1, 0, \dots, 0) = x(t)$$

$$\varphi_t(e_i(0)) = e_i(t)$$

preserves $\langle \cdot, \cdot \rangle$, so $\varphi_t \in SO^+(1, n)$. But $\varphi_t(1, 0, \dots, 0) = x$
 by construction, so $SO^+(1, n)$ is transitive on H^n .

The isotropy subgroup of p is

$$\left\{ \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \boxed{A} & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix} \mid A \in SO(n) \right\} \cong SO(n)$$

and this acts on $T_p H^n \cong \mathbb{R}^n$ in the usual way, $v \mapsto Av$.

Since $SO(n)$ acts transitively on 2-planes in \mathbb{R}^n , the isotropy
 subgroup of p acts transitively on 2-planes in $T_p H^n$.

It follows that $K(p) = \text{const} \forall \sigma \subset T_x H^n \forall x \in H^n$.

By problem #3, this constant is -1 .