

Math 621 HW #8 - Outline of Solutions

Note Title

4/6/2018

1. (a) Fix $p \in M$, and let $B_\epsilon(p)$ be a normal ball. By homogeneity, $B_\epsilon(q)$ is a normal ball $\forall q \in M$. Thus any geodesic with $\gamma(0) = q$ and speed s can be defined for $t \in (-\epsilon/s, \epsilon/s)$ independent of q . It follows that any geodesic can be extended to all of \mathbb{R} .

(b) Lie groups with left invariant metrics are homogeneous. By Hopf-Rinow, if $g \in G$, \exists geodesic γ with $\gamma(0) = e$, $\gamma(t_0) = g$. Then $\gamma(t) = \exp_e(tv)$ where $v = \gamma'(0)$ so $g = \exp_e(t_0 v)$.

(c) Write $e^M = 1 + M + \frac{M^2}{2!} + \dots$ instead of $\exp(M)$, for clarity. If $SL(2, \mathbb{R})$ had a biinvariant metric, then geodesics through e with respect to this metric would be 1-parameter subgroups, so all such geodesics would be of the form

$$\gamma(t) = e^{tM} \quad \text{for some } M \in \mathfrak{sl}(2, \mathbb{R}).$$

Thus by (b), it suffices to show that there is an element of $SL(2, \mathbb{R})$ that isn't equal to e^M for any $M \in \mathfrak{sl}(2, \mathbb{R})$.

Claim: $\begin{bmatrix} -1 & \\ 0 & -1 \end{bmatrix} \neq e^M$ for any $M \in \mathfrak{sl}(2, \mathbb{R})$.

Pf: Put M in Jordan canonical form, $M = P \begin{bmatrix} \lambda_1 & * \\ 0 & \lambda_2 \end{bmatrix} P^{-1}$ for some $P \in GL(2, \mathbb{C})$, $* = 0$ or 1 . If $* = 0$ then M is diagonalizable, so e^M is as well. Thus $* = 1$ and so $\lambda_1 = \lambda_2 = 0$ since $\text{tr } M = 0$.

$$\text{Then } e^M = P e^{\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}} P^{-1} = P \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} P^{-1}$$

and this can't be $\begin{bmatrix} -1 & \\ 0 & -1 \end{bmatrix}$ (because of its eigenvalues).

Alternate solution to (c) that doesn't use (b): Consider $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathfrak{sl}(2, \mathbb{R})$, $g = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \in SL(2, \mathbb{R})$. Then $(L_g)_* (P_{g^{-1}})_* X = gXg^{-1} = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} = 4X$ so a biinvariant metric would satisfy $|X|^2 = |(L_g)_* (P_{g^{-1}})_* X|^2 = |4X|^2 = 16|X|^2$.

2. From HW 7 #1 (b), we have

$$\nabla_x R(y,z)W = \nabla_x (R(y,z)W) - R(\nabla_x y, z)W - R(y, \nabla_x z)W - R(y,z)\nabla_x W.$$

Use \sum_{cyc} to denote a sum over the three cyclic permutations of (x, y, z) :

$(x, y, z), (y, z, x), (z, x, y)$. Then:

$$\begin{aligned} \sum_{\text{cyc}} \nabla_x R(y,z)W &= \sum_{\text{cyc}} \nabla_x (R(y,z)W) - \sum_{\text{cyc}} R(\nabla_x y, z)W - \sum_{\text{cyc}} R(y, \nabla_x z)W \\ &\quad - \underbrace{\left(\sum_{\text{cyc}} R(\nabla_x y, z)W + \sum_{\text{cyc}} R(y, \nabla_x z)W \right)}. \end{aligned}$$

Now

$$\begin{aligned} \textcircled{1} &= \sum_{\text{cyc}} \left(\nabla_x \nabla_z \nabla_y W - \nabla_x \nabla_y \nabla_z W + \nabla_x \nabla_{[y,z]} W \right) \\ &\quad + \sum_{\text{cyc}} \left(-\nabla_z \nabla_y \nabla_x W + \nabla_y \nabla_z \nabla_x W - \nabla_{[y,z]} \nabla_x W \right) \end{aligned}$$

$$= \sum_{\text{cyc}} \left(R(x, [y,z])W - \nabla_{[x, [y,z]]} W \right)$$

$$= \sum_{\text{cyc}} R(x, [y,z])W \quad \text{by Jacobi}$$

while

$$\textcircled{2} = \sum_{\text{cyc}} R(\nabla_x y, z)W + \sum_{\text{cyc}} R(z, \nabla_y x)W$$

$$= - \sum_{\text{cyc}} R(z, [x, y])W$$

and so $\sum_{\text{cyc}} \nabla_x R(y,z)W = \textcircled{1} + \textcircled{2} = 0$, as desired.

2. For the (0,4)-tensor version, note

Case.

$$X \langle R(Y, Z)W, T \rangle = \langle \nabla_X (R(Y, Z)W), T \rangle + \langle R(Y, Z)W, \nabla_X T \rangle$$

$$\begin{aligned} (\nabla_X R)(Y, Z, W, T) &= X \langle R(Y, Z)W, T \rangle - R(\nabla_X Y, Z, W, T) - R(Y, \nabla_X Z, W, T) \\ &\quad - R(Y, Z, \nabla_X W, T) - R(Y, Z, W, \nabla_X T) \\ &= \langle \nabla_X (R(Y, Z)W), T \rangle - R(\nabla_X Y, Z, W, T) - R(Y, \nabla_X Z, W, T) \\ &\quad - R(Y, Z, \nabla_X W, T) \\ &= \langle \nabla_X R(Y, Z)W, T \rangle \end{aligned}$$

and so the (0,4) version follows immediately from the (1,3) version.

3. (a) $\nabla_X X = \nabla_Y Y = \nabla_{X+Y}(X+Y) = 0 \Rightarrow 0 = \nabla_X Y + \nabla_Y X = 2\nabla_X Y - [X, Y]$
 $\Rightarrow \nabla_X Y = \frac{1}{2}[X, Y].$

(b) Use (a) and Jacobi identity.

(c) $K(\sigma) = R(X, Y, X, Y) = \frac{1}{4} \langle [[X, Y], X], Y \rangle \stackrel{\text{Bivariance}}{=} -\frac{1}{4} \langle [X, Y], [Y, X] \rangle = \frac{1}{4} \|[X, Y]\|^2.$

(d) If we view M as a submanifold of \mathbb{R}^3 , then $\partial_x = (1, 0, F_x)$
 and $\partial_y = (0, 1, F_y)$ so the matrix for g is as claimed.

Straightforward calculation (from the coord. formulas for Γ_{ij}^k):

$$\begin{aligned} \Gamma_{xx}^x &= \frac{1}{(1+F_x^2+F_y^2)} F_x F_{xx} & \Gamma_{xy}^x &= \Gamma_{yx}^x = \frac{1}{(1+F_x^2+F_y^2)} F_x F_{xy} & \Gamma_{yy}^x &= \frac{1}{(1+F_x^2+F_y^2)} F_x F_{yy} \\ \Gamma_{xx}^y &= \frac{1}{(1+F_x^2+F_y^2)} F_y F_{xx} & \Gamma_{xy}^y &= \Gamma_{yx}^y = \frac{1}{(1+F_x^2+F_y^2)} F_y F_{xy} & \Gamma_{yy}^y &= \frac{1}{(1+F_x^2+F_y^2)} F_y F_{yy} \end{aligned}$$

$$\Rightarrow R_{xyx}^x = \frac{1}{(1+F_x^2+F_y^2)^2} F_x F_y (F_{xy}^2 - F_{xx} F_{yy}), \quad R_{xyx}^y = \frac{1}{(1+F_x^2+F_y^2)^2} (F_{xx} F_{yy} - F_{xy}^2)$$

$\Rightarrow K = \text{as given.}$

(e) Gaussian curvature at $(0, 0, a)$ is routine.

Since $\{x^2 + y^2 + z^2 = a^2\}$ is homogeneous, it has the same curvature at every point, so $K = \frac{1}{a^2}$ everywhere.