

Math 621 HW 7 - Outline of Solutions

Note Title

3/28/2018

1. (a) Uniqueness:

∇ is given for $(1,0)$ tensors. The properties imply:

• for $(0,0)$ tensors: if $f \in C^\infty(M)$ then

$$(\nabla_X f) Y + f \nabla_X Y = \nabla_X(fY) = (Xf)Y + f \nabla_X Y$$

for all Y , so: (1) $\nabla_X f = Xf$.

• for $(0,1)$ tensors: if $\alpha \in \Gamma(T_1^0(M))$ then

$$\begin{aligned} X\alpha(Y) &= \nabla_X(\alpha(Y)) = \nabla_X(c(Y \otimes \alpha)) = c \nabla_X(Y \otimes \alpha) \\ &= c(\nabla_X Y \otimes \alpha + Y \otimes \nabla_X \alpha) = \alpha(\nabla_X Y) + (\nabla_X \alpha)(Y) \end{aligned}$$

so $\nabla_X \alpha$ is the $(0,1)$ tensor satisfying

$$(2) \quad \nabla_X \alpha(Y) = X\alpha(Y) - \alpha(\nabla_X Y). \quad (\text{easy to check: this is a tensor})$$

Finally, ∇_X is determined on (p,q) tensors by:

$$(3) \quad \nabla_X(X_1 \otimes \dots \otimes X_p \otimes \alpha_1 \otimes \dots \otimes \alpha_q) = \sum_{i=1}^p X_i \otimes \dots \otimes (\nabla_X X_i) \otimes \dots \otimes X_p \otimes \alpha_1 \otimes \dots \otimes \alpha_q \\ + \sum_{i=1}^q X_1 \otimes \dots \otimes X_p \otimes \alpha_1 \otimes \dots \otimes (\nabla_X \alpha_i) \otimes \dots \otimes \alpha_q$$

where $X_i \in \text{Vect}(M)$ and $\alpha_i \in \Omega^1(M)$.

Note: Writing a (p,q) tensor as (a sum of) $X_1 \otimes \dots \otimes X_p \otimes \alpha_1 \otimes \dots \otimes \alpha_q$ might implicitly assume that we're working locally; but (i) implies that ∇_X is local, so this is ok.

Existence: Define ∇_X using (1), (2), (3). Then almost by definition it satisfies properties (i) and (ii) (the latter since (ii) holds when S is a $(1,1)$ tensor by construction).

1. (b) Suppose $S = \alpha_1 \otimes \dots \otimes \alpha_g$ where $\alpha_i \in \Omega^1(M)$. Then
Cont'd.

$$\begin{aligned} (\nabla_x S)(X_1, \dots, X_g) &= \sum (\alpha_1 \otimes \dots \otimes (\nabla_x \alpha_i) \dots \otimes \alpha_g)(X_1, \dots, X_g) \\ &= \sum \alpha_1(X_1) \dots (\nabla_x \alpha_i(X_i) - \alpha_i(\nabla_x X_i)) \dots \alpha_g(X_g) \\ &= X(\alpha_1(X_1) \dots \alpha_g(X_g)) - \sum \alpha_1(X_1) \dots \alpha_i(\nabla_x X_i) \dots \alpha_g(X_g) \end{aligned}$$

and the result follows for $(0, g)$ tensors.

For $(1, g)$ tensors the formula is:

$$\boxed{(\nabla_x S)(X_1, \dots, X_g) = \nabla_x (S(X_1, \dots, X_g)) - \sum_{i=1}^g S(X_1, \dots, \nabla_x X_i, \dots, X_g)}.$$

To see this, suppose $S = Y \otimes T$ where $Y \in \text{Vect}(M)$ and $T = (0, g)$ tensor.

Then $\nabla_x S = (\nabla_x Y) \otimes T + Y \otimes (\nabla_x T)$ so

$$\begin{aligned} (\nabla_x S)(X_1, \dots, X_g) &= T(X_1, \dots, X_g) \nabla_x Y + (X(T(X_1, \dots, X_g)) - \sum_i T(\dots \nabla_x X_i \dots)) Y \\ &= \nabla_x (T(X_1, \dots, X_g) Y) - \sum_i T(\dots \nabla_x X_i \dots) Y \\ &= \nabla_x (S(X_1, \dots, X_g)) - \sum_i S(\dots \nabla_x X_i \dots). \end{aligned}$$

2. (a) From the formula for $L_x S$ from HW4 #3(c),

$$\begin{aligned} (L_x g)(Y, Z) &= Xg(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]) \\ &= g(\nabla_x Y, Z) + g(Y, \nabla_x Z) - g(\nabla_x Y - \nabla_Y X, Z) - g(Y, \nabla_x Z - \nabla_Z X) \\ &= g(\nabla_Y X, Z) + g(\nabla_Z X, Y). \quad \square \end{aligned}$$

2. (4) Consider a point $g \in S_a(p)$. Write $g = \exp_p v$ for some $v \in T_p M$, $|v| = a$.
 Case. Let γ denote the geodesic $t \mapsto \exp_p(tv)$. Since $g \mapsto \varphi(s, g)$

is an isometry for fixed s , the image of γ under this map is also a geodesic,
 $t \mapsto \varphi(s, \exp_p(tv)) =: \gamma_s(t)$.

Note $\gamma_s(0) = \varphi(s, p) = p$ and γ_s has the same speed as γ , so
 $\gamma_s(t) = \exp_p(t\nu(s))$ for some $\nu(s) \in T_p M$, $|\nu(s)| = a$
 (Specifically $\nu(s) = \frac{d}{ds} \big|_{t=0} \gamma_s(t)$).

Thus $\gamma_s(1) \in S_a(p) \forall s$. Since $\gamma_s(1) = \varphi(s, g)$ and
 $X(g) = \frac{d}{ds} \big|_{s=0} \varphi(s, g)$, $X(g)$ is tangent to $S_a(p)$.

(c) $\varphi(t, g) = \text{local flow for } X \text{ on } M \Rightarrow f \circ \varphi(t, f^{-1}(g)) = \text{local flow for } Y \text{ on } N$.

Write $\varphi_t: M \rightarrow M$, $\psi_t: N \rightarrow N$ for $\varphi_t(g) = \varphi(t, g)$, $\psi_t(g) = f \circ \varphi(t, f^{-1}(g))$.

Then $f \circ \varphi_t = \psi_t \circ f$, so for $v, w \in T_p M$, if ψ_t is an isometry then

$$\begin{aligned} \langle (\varphi_t)_* v, (\varphi_t)_* w \rangle_M &= \langle (f \circ \varphi_t)_* v, (f \circ \varphi_t)_* w \rangle_N = \langle (\psi_t \circ f)_* v, (\psi_t \circ f)_* w \rangle_N \\ &= \langle f_* v, f_* w \rangle_N = \langle v, w \rangle_M \end{aligned}$$

And φ_t is an isometry. Conversely, interchange $M \leftrightarrow N$ and replace f by f^{-1} .

(d) If X is Killing, then from (a), $g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0$.

Conversely, write $\varphi_t = \text{time } t \text{ flow of } X$. Since $\varphi_{t+t'} = \varphi_t \circ \varphi_{t'}$, Lemma \Rightarrow

$$\forall t, \quad \frac{d}{dt} ((\varphi_t^* g)(Y, Z))(p) = g(\nabla_{(\varphi_t)_* Y} X, (\varphi_t)_* Z) + g((\varphi_t)_* Y, \nabla_{(\varphi_t)_* Z} X).$$

If $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0 \forall Y, Z$, then $\frac{d}{dt} ((\varphi_t^* g)(Y, Z)) = 0 \forall t$

So $(\varphi_t^* g)(Y, Z) = g(Y, Z)$ and X is Killing.

3. Solution 1 Equip $T_p M$ with the pullback metric $(\exp_p)^* g$ (note: not the flat metric on \mathbb{R}^n), so that \exp_p is an isometry. Let x^1, \dots, x^n be the standard coordinates on $T_p M = \mathbb{R}^n$. Note at all points in $T_p M$, $\partial_1, \dots, \partial_n$ is a basis for the tangent space to $T_p M$. We can assume that at $0 \in T_p M$, $\partial_1, \dots, \partial_n$ is actually an orthonormal basis of $T_0(T_p M) = T_p M$; if not, just change coords x^i by applying a suitable invertible linear transformation to the vector space $T_p M$ (= diffeo of $T_p M$).

Now define linearly independent vector fields E_1, \dots, E_n in a normal ball $U = B_r(p)$ by $E_i := (\exp_p)_* (\partial_i)$. Since $(\exp_p)_*(0) = \text{identity}$, $\{E_1(p), \dots, E_n(p)\}$ is an orthonormal basis of $T_p M$.

Claim $\nabla_{E_i} E_j(p) = 0 \forall i, j$.

PF Any straight line $t \mapsto (tv^1, \dots, tv^n)$ in $T_p M$ through 0 (for fixed v^1, \dots, v^n) is the pullback of a geodesic in M through p , so each of these lines is a geodesic in $B(0) \subset T_p M$.

$$\Rightarrow \frac{d^2 x^k}{dt^2} + \Gamma_{ij}^k \frac{dx^i}{dt} \frac{dx^j}{dt} = 0 \text{ where } \Gamma = \text{Christoffel symbols on } T_p M \xrightarrow{t=0} \Gamma_{ij}^k(0) v^i v^j = 0.$$

Since this holds $\forall v^i, v^j$, $\Gamma_{ij}^k(0) = 0 \forall i, j, k$, so $\nabla_i \partial_j(0) = 0 \forall i, j$.

Push forward to U to get $\nabla_{E_i} E_j(p) = 0 \forall i, j$. \square
 (using the fact that \exp_p is an isometry!)

Now we have vector fields E_1, \dots, E_n with $\nabla_{E_i} E_j(p) = 0 \forall i, j$.

Use Gram-Schmidt to make E_1, \dots, E_n orthonormal throughout U (note they're still smooth by the G-S construction). We claim this doesn't affect $\nabla_{E_i} E_j(p) = 0$, and then we're done. Gram-Schmidt consists of steps replacing E_i by

$\frac{1}{\langle E_i, E_i \rangle} E_i$ or by $E_i := \frac{\langle E_i, E_k \rangle}{\langle E_k, E_k \rangle} E_k := E_i - f_{ik} E_k$. Let's consider the latter (the former is similar). Then $\nabla_{E_i - f_{ik} E_k} E_j(p) = 0$ by linearity, while

$$\nabla_{E_j} (E_i - f_{ik} E_k)(p) = \underbrace{\nabla_{E_j} E_i(p)}_0 - f_{ik} \underbrace{\nabla_{E_j} E_k(p)}_0 - E_j(f_{ik})(p) E_k(p) = 0$$

since $E_j \langle E_i, E_k \rangle(p) = \langle \nabla_{E_j} E_i, E_k \rangle(p) + \langle E_i, \nabla_{E_j} E_k \rangle(p) = 0$ and similarly $E_j \langle E_k, E_k \rangle(p) = 0$.

3. Solution 2 not using the hint.

cont'd. Choose any orthonormal frame $E_1(p), \dots, E_n(p)$ for $T_p M$.

For any q in a normal ball around p , we can join p to q along a radial geodesic γ , and define $E_i(q) \in T_q M$ to be the parallel transport of $E_i(p)$ along γ . Since parallel transport preserves $\langle \cdot, \cdot \rangle$, $\{E_i(q)\}$ is an orthonormal frame for $T_q M$.

Also, if we define $\gamma_i =$ geodesic with initial condition $(p, E_i(p))$, then E_j is parallel along γ_i : by construction $\Rightarrow \nabla_{E_i} E_j(p) = 0$.

However: one has to prove that the vector fields E_i constructed from $E_i(p)$ this way ("by radial parallel transport") are actually smooth. This is true but requires a bit of work:

Lemma Let $V(p) \in T_p M$, and let V be the vector field (in a normal ball around p) constructed by radial parallel transport from $V(p)$.

Then V is smooth.

PF: Identify coords on M with coords x^1, \dots, x^n on $T_p M$ by exp.

For fixed x^1, \dots, x^n , $V(x^1, \dots, x^n) = V^k(t) \partial_k$ where

$$(*) \quad \frac{d}{dt} V^k(t) = -V^j(t) x^i \Gamma_{ij}^k(t, x^1, \dots, x^n), \quad V^k(0) = V^k.$$

We can now let x^1, \dots, x^n vary and write $V^k(t) = V^k(x^1, \dots, x^n, t)$.

Introduce auxiliary functions $w^1(t), \dots, w^n(t)$. Then we can write (*) as

$$(**) \quad \begin{cases} \frac{d}{dt} V^k(t) = -V^j(t) w^i(t) \Gamma_{ij}^k(t, w^1(t), \dots, w^n(t)) \\ \frac{d}{dt} w^k(t) = 0 \end{cases}$$

with initial conditions $V^k(0) = V^k$, $w^k(0) = x^k$. The (**) is a system of 1st order ODEs whose solutions vary smoothly in both t and the initial conditions — so $V^k(t)$ depends smoothly on x^1, \dots, x^n . \square