

Math 621 HW#5 - Outline of Solutions

Note Title

3/7/2018

1. Both sides of $\mathcal{L}_X = d_{ix} + i_x d$ are local operators, so it suffices to check in a chart. Note

$$\mathcal{L}_X f = Xf = (df)(X) = i_X df = (d_{ix} + i_x d)f$$

and

$$\mathcal{L}_X(dx_i) = d\mathcal{L}_X(x_i) = dX(x_i) = d(dx_i(X)) = d(i_X dx_i) = (d_{ix} + i_x d)dx_i.$$

Now suppose Cartan's magic formula holds for $\omega_1 \in \Omega^k$ and $\omega_2 \in \Omega^l$. Then

$$\mathcal{L}_X(\omega_1 \wedge \omega_2) = (\mathcal{L}_X \omega_1) \wedge \omega_2 + \omega_1 \wedge (\mathcal{L}_X \omega_2)$$

(Check how both sides evaluate on vector fields X_1, \dots, X_{k+l-1}) $\rightarrow i_X(\omega_1 \wedge \omega_2) = (i_X \omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge (i_X \omega_2)$

and we find that

$$\begin{aligned} (d_{ix} + i_x d)(\omega_1 \wedge \omega_2) &= ((d_{ix} + i_x d)\omega_1) \wedge \omega_2 + \omega_1 \wedge ((d_{ix} + i_x d)\omega_2) \\ &= \mathcal{L}_X \omega_1 \wedge \omega_2 + \omega_1 \wedge \mathcal{L}_X \omega_2 = \mathcal{L}_X(\omega_1 \wedge \omega_2). \end{aligned}$$

2-(a) Induction on k : for $k=0$, this is $df(X) = X(f)$. For general k ,

$$\begin{aligned} d\omega(X_0, \dots, X_k) &= i_{X_0} d\omega(X_1, \dots, X_k) \\ &= \mathcal{L}_{X_0} \omega(X_1, \dots, X_k) - d_{i_{X_0}} \omega(X_1, \dots, X_k) \\ &= X_0 \omega(X_1, \dots, X_k) - \omega([X_0, X_1], \dots, X_k) - \dots - \omega(X_1, \dots, [X_0, X_k]) \\ &\quad - d(i_{X_0} \omega)(X_1, \dots, X_k) \end{aligned}$$

And now use the induction hypothesis on $i_{X_0} \omega$.

(b) Straightforward check that the right hand side is antisymmetric

$$(d\omega(\dots, X_i, X_{i+1}, \dots)) = -d\omega(\dots, X_{i+1}, X_i, \dots).$$

One also has to check that $d\omega$ as defined is a tensor.

$$fx_i \omega(\dots, \hat{x}_i, \dots) + (x_i f) \omega(\dots, \hat{x}_i, \dots)$$

2. (b) $d\omega(\dots, fX_k, \dots) = \sum_{i \neq k} (-1)^i X_i (f\omega(x_0, \dots, \hat{x}_i, \dots, x_n)) + (-1)^k X_k (\omega(\dots, \hat{x}_k, \dots))$

Cont'd.

$$+ \sum_{i < j, i \neq k, j \neq k} (-1)^{i+j} f\omega([x_i, x_j], x_0, \dots, \hat{x}_k, \dots, \hat{x}_j, \dots, x_n)$$

$$+ \sum_{j > k} (-1)^{k+j} \underbrace{\omega([fx_k, x_j], \dots, \hat{x}_k, \dots, \hat{x}_j, \dots)}_{f\omega([x_k, x_j], \dots, \hat{x}_k, \dots, \hat{x}_j, \dots)} - (x_i f) \omega(x_k \dots \hat{x}_k \dots \hat{x}_j \dots)$$

$$+ \sum_{i < k} (-1)^{i+k} \underbrace{\omega([x_i, fx_k], \dots, \hat{x}_i, \dots, \hat{x}_k, \dots)}_{f\omega([x_k, x_i], \dots, \hat{x}_i, \dots, \hat{x}_k, \dots)} - (x_i f) \omega(x_k \dots \hat{x}_i \dots \hat{x}_j \dots)$$

$$= f d\omega(\dots, X_k, \dots) \text{ because the red terms cancel.}$$

(c) $\omega = f \in \Omega^0 M$:

$$\begin{aligned} d^L f(X_0, X_1) &= X_0 df(X_1) - X_1 df(X_0) - df([X_0, X_1]) \\ &= X_0 X_1 f - X_1 X_0 f - [X_0, X_1] f = 0. \end{aligned}$$

$\omega \in \Omega^1 M$:

$$\begin{aligned} d^L \omega(X_0, X_1, X_2) &= X_0 d\omega(X_1, X_2) - X_1 d\omega(X_0, X_2) + X_2 d\omega(X_0, X_1) \\ &\quad - d\omega([X_0, X_1], X_2) + d\omega([X_0, X_2], X_1) - d\omega([X_1, X_2], X_0). \end{aligned}$$

Now use $d\omega(X_0, X_1) = X_0 \omega(X_1) - X_1 \omega(X_0) - \omega([X_0, X_1])$ and Jacobi to get $d^L \omega(X_0, X_1, X_2) = 0$.

3. (a) Note $\varphi = \varphi^{-1}$ so φ is smooth with smooth inverse. We also need to check:

$\varphi(D^n) \subset H^n$: if $x \in D^n$ then the first coordinate of $\varphi(x)$ is

$$-1 + \frac{2(x_1 + 1)}{\|x - p\|^2} \geq -1 + \frac{2(x_1 + 1)}{(x_1 + 1)^2} = -1 + \frac{2}{x_1 + 1} \geq -1 + \frac{2}{2} = 0$$

$\varphi(H^n) \subset D^n$: if $x \in H^n$ then

$$\|\varphi(x)\|^2 = \left(p + \frac{2(x-p)}{\|x-p\|^2}\right) \cdot \left(p + \frac{2(x-p)}{\|x-p\|^2}\right) = p \cdot p + \frac{4(x-p) \cdot p}{\|x-p\|^2} + \frac{4}{\|x-p\|^2} \leq p \cdot p = 1.$$

Thus $\varphi: D^n \rightarrow H^n$ is a diffeo.

3. (b) Translate everything by p : let $\psi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be defined by $\psi(x) = x + p$,
 Cont'd. and consider $(D^n + (-p), \tilde{g}) \xrightarrow{\tilde{\psi}} (H^n + (-p), \tilde{h})$ where $D^n + (-p) = \{x - p \mid x \in D^n\}$ etc.
 $\downarrow \quad \downarrow$
 $(D^n, g) \xrightarrow{\psi} (H^n, h)$

$$\tilde{\psi} = \psi^{-1} \circ \psi \circ \tilde{\psi}, \tilde{g} = \psi^* g, \tilde{h} = \psi^* h.$$

It suffices to show $\tilde{\psi}^* \tilde{h} = \tilde{g}$ ($\Rightarrow \psi^* h = (\psi \circ \tilde{\psi})^* h = (\psi)^* \tilde{\psi}^* \psi^* h = g$).

Given $D^n + (-p)$ coordinates x_1, \dots, x_n and $H^n + (-p)$ coordinate y_1, \dots, y_n .

$$\Rightarrow \tilde{g}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{4\delta_{ij}}{(1 - \|x + p\|^2)^2}, \quad \tilde{h}\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = \frac{\delta_{ij}}{(y_i - 1)^2}.$$

$$\tilde{\psi}(x) = \frac{2x}{\|x\|^2} \Rightarrow y_j = \frac{2x_j}{\|x\|^2} \quad \forall j.$$

$$\Rightarrow \tilde{\psi}_* \frac{\partial}{\partial x_i} = \sum_k \frac{\partial y_k}{\partial x_i} \frac{\partial}{\partial y_k} = \frac{2}{\|x\|^2} \frac{\partial}{\partial y_i} - \sum_k \frac{4x_i x_k}{\|x\|^4} \frac{\partial}{\partial y_k}$$

$$\Rightarrow (\tilde{\psi}^* \tilde{h})\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \tilde{h}\left(\frac{2}{\|x\|^2} \frac{\partial}{\partial y_i} - \sum_k \frac{4x_i x_k}{\|x\|^4} \frac{\partial}{\partial y_k}, \frac{2}{\|x\|^2} \frac{\partial}{\partial y_j} - \sum_l \frac{4x_j x_l}{\|x\|^4} \frac{\partial}{\partial y_l}\right)$$

$$= \frac{1}{(y_i - 1)^2} \left[\frac{4}{\|x\|^4} \delta_{ij} - \frac{8x_i x_j}{\|x\|^6} - \frac{8x_i x_j}{\|x\|^6} + \underbrace{\sum_k \frac{16x_i x_j x_k}{\|x\|^8}}_{\frac{16x_i x_j}{\|x\|^6}} \right]$$

$$\begin{aligned} y_i - 1 &= 2x_i - \|x\|^2 \\ &= 2x_i - x_1^2 - x_2^2 - \dots \\ &= 1 - (x_i - 1)^2 - x_2^2 - \dots \\ &= 1 - \|x + p\|^2 \end{aligned}$$

$$= \frac{4}{(y_i - 1)^2 \|x\|^4} \delta_{ij}$$

$$= \frac{4\delta_{ij}}{(-\|x + p\|^2)^2} = \tilde{g}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

as desired.

4. Lemma Y left invariant, $\gamma(t) = \text{path in } G$, $X \in g$ defined by
 $X = \frac{d}{dt} \Big|_{t=t_0} (\gamma(t) \gamma(t)^{-1})$ for fixed t_0 . Then
 $\frac{d}{dt} \Big|_{t=t_0} \text{Ad}(\gamma(t))(Y) = [X, \text{Ad}(\gamma(t_0))Y].$

PF $R_{\gamma(t)^{-1}} = R_{\gamma(t_0) \gamma(t_0)^{-1}} R_{\gamma(t_0)^{-1}} \Rightarrow$

$$\text{Ad}(\gamma(t))Y = (R_{\gamma(t)^{-1}})_* Y = (R_{\gamma(t_0) \gamma(t_0)^{-1}})_* (R_{\gamma(t_0)^{-1}})_* Y = (R_{\gamma(t_0) \gamma(t_0)^{-1}})_* \text{Ad}(\gamma(t_0))Y.$$

Now if φ_t = local flow of X then

$$\begin{aligned} & \frac{d}{dt} \Big|_{t=t_0} \gamma(t) \gamma(t)^{-1} = -X = \frac{d}{dt} \Big|_{t=t_0} \varphi_t(e) \\ \text{So } & \frac{d}{dt} \Big|_{t=t_0} \text{Ad}(\gamma(t))Y = \frac{d}{dt} \Big|_{t=t_0} (R_{\gamma(t) \gamma(t)^{-1}})_* \text{Ad}(\gamma(t_0))Y \\ &= \frac{d}{dt} \Big|_{t=t_0} (R_{\varphi_t(e)})_* \text{Ad}(\gamma(t_0))Y \\ &= \frac{d}{dt} \Big|_{t=t_0} \text{Ad}(\varphi_t(e)) \text{Ad}(\gamma(t_0))Y \\ &= [X, \text{Ad}(\gamma(t_0))Y]. \quad \square \end{aligned}$$

Main Proof Let $h \in G$ and let Y, Z be left invt vector fields.

It suffices to prove

$$\langle Y, Z \rangle_g = \langle (R_h)_* Y, (R_h)_* Z \rangle_{gh} \quad \forall g \in G.$$

Since $Y, Z, (R_h)_* Y, (R_h)_* Z$ are left invariant, $\langle Y, Z \rangle_g = \langle Y, Z \rangle_e$ and

$$\langle (R_h)_* Y, (R_h)_* Z \rangle_{gh} = \langle (R_h)_* Y, (R_h)_* Z \rangle_e.$$

Now let $\gamma: [0, 1] \rightarrow G$ be a path with $\gamma(0) = e$, $\gamma(1) = h^{-1}$, and define

$$f(t) = \langle (R_{\gamma(t)^{-1}})_* Y, (R_{\gamma(t)^{-1}})_* Z \rangle_e = \langle \text{Ad}(\gamma(t))Y, \text{Ad}(\gamma(t))Z \rangle, \quad 0 \leq t \leq 1.$$

For $t_0 \in [0, 1]$, we have

$$\frac{d}{dt} \Big|_{t=t_0} f(t) = \frac{d}{dt} \Big|_{t=t_0} \langle \text{Ad}(\gamma(t))Y, \text{Ad}(\gamma(t))Z \rangle + \langle \text{Ad}(\gamma(t_0))Y, \frac{d}{dt} \Big|_{t=t_0} \text{Ad}(\gamma(t))Z \rangle$$

$$\begin{aligned} \text{by Lemma; } & \longrightarrow = \langle [X, \text{Ad}(\gamma(t_0))Y], \text{Ad}(\gamma(t_0))Z \rangle + \langle \text{Ad}(\gamma(t_0))Y, [X, \text{Ad}(\gamma(t_0))Z] \rangle \\ X = & \text{as in lemma} \\ & \Rightarrow \text{by assumption.} \end{aligned}$$

Thus $\langle Y, Z \rangle_e = f(0) = f(1) = \langle (R_h)_* Y, (R_h)_* Z \rangle_e$ as desired.

— alternate solution on next page —

4. Alternate Solution using the fact that $\{\exp X\}$ generates G :

Cont'd. Note $\text{Ad}_{\exp(tx)} = \text{Ad}_{\exp((t-t_0)x)} \text{Ad}_{\exp(t_0x)}$ for any t, t_0 : so $\forall t_0$,

$$\frac{d}{dt} \Big|_{t=t_0} \text{Ad}_{\exp(tx)} Y = \frac{d}{dt} \Big|_{t=t_0} \text{Ad}_{\exp(t_0x)} (\text{Ad}_{\exp(t_0x)} Y) = [X, \text{Ad}_{\exp(t_0x)} Y]$$

$$\begin{aligned} \Rightarrow \frac{d}{dt} \Big|_{t=t_0} \langle \text{Ad}_{\exp(tx)} Y, \text{Ad}_{\exp(tx)} Z \rangle &= \langle [X, \text{Ad}_{\exp(t_0x)} Y], \text{Ad}_{\exp(t_0x)} Z \rangle \\ &\quad + \langle \text{Ad}_{\exp(t_0x)} Y, [X, \text{Ad}_{\exp(t_0x)} Z] \rangle \\ &= 0. \end{aligned}$$

Thus $\langle \text{Ad}_{\exp(t_0x)} Y, \text{Ad}_{\exp(t_0x)} Z \rangle$ is constant in t

$$\Rightarrow \langle Y, Z \rangle_c = \langle \text{Ad}_{\exp(X)} Y, \text{Ad}_{\exp(X)} Z \rangle_c = \langle (R_{\exp(X)})^* Y, (R_{\exp(X)})^* Z \rangle_{\exp(X)}.$$

Since $\{\exp X\}$ generates G , it follows that

$$\langle Y, Z \rangle_c = \langle (R_g)^* Y, (R_g)^* Z \rangle_g \quad \forall g \in G$$

and then that $\langle \cdot, \cdot \rangle$ is right invariant.

5. (a) For any path $M(t)$ in $SO(n)$ with $M(0) = \mathbf{1}$,

$$(M(t))^T M(t) = \mathbf{1} \Rightarrow (M'(t))^T M(t) + (M(t))^T M'(t) = \mathbf{0} \Rightarrow (M'(0))^T = -M'(0),$$

thus $SO(n) = T_{\mathbf{1}} SO(n) \subset \{\text{Skew symmetric } n \times n \text{ matrices}\}$.

Now $\dim \{\text{Skew symmetric } n \times n \text{ matrices}\} = \frac{n(n-1)}{2}$ by linear algebra,

and $\dim SO(n) = \frac{n(n-1)}{2}$ as well (for example, one can prove this by induction on n , using the fact that the map $SO(n) \rightarrow S^{n-1}$ given by mapping to the first row has fiber $SO(n-1)$).

$$\Rightarrow SO(n) = \{\text{Skew-symmetric } n \times n \text{ matrices}\}.$$

S. (a) Similarly if $M(t)$ is a path in $SL(n, \mathbb{R})$ with $M(0)=\mathbb{1}$ then
 cont'd.

$$0 = \frac{d}{dt} \Big|_{t=0} \det M(t) = \text{tr } M'(0)$$

several ways to see this, e.g. write $\det M(t)$ explicitly as a sum over permutations

So $sl(n, \mathbb{R}) \subset \{\text{traceless } n \times n \text{ matrices}\}$.

Since $\dim SL(n, \mathbb{R}) = n^2 - 1 = \dim \{\mathcal{T}\}$, we get $sl(n, \mathbb{R}) = \{\text{traceless } n \times n \text{ matrices}\}$.

(b) If $N(t)$ is a path in G with $N(0)=\mathbb{1}$ then

$$\text{Ad}(M) (N'(0)) = \frac{d}{dt} \Big|_{t=0} M \cdot N(t) \cdot M^{-1} = M \cdot N'(0) \cdot M^{-1}.$$

(c) If φ_t = local flow of X then

$$\begin{aligned} [X, Y] &= \frac{d}{dt} \Big|_{t=0} \text{Ad}(\varphi_t(e)) Y = \frac{d}{dt} \Big|_{t=0} (\varphi_t(e) \cdot Y \cdot \varphi_{-t}(e)) \\ &= \left(\frac{d}{dt} \Big|_{t=0} \varphi_t(e) \right) \cdot Y + Y \cdot \left(\frac{d}{dt} \Big|_{t=0} \varphi_{-t}(e) \right) \\ &= X \cdot Y - Y \cdot X. \end{aligned}$$

(d) Symmetric: $\text{tr}(Y^T X) = \text{tr}((X^T Y)^T) = \text{tr}(X^T Y)$.

Positive definite: if $X = (a_{ij})$ then $\text{tr}(X^T X) = \sum_{i,j} a_{ij}^2 \geq 0$,
 with equality $\Leftrightarrow X = 0$.

(e) For $X, Y, Z \in so(n)$, $(X^T = -X, \text{etc.})$

$$\begin{aligned} \langle [X, Y], Z \rangle_e &= \text{tr}([X, Y]^T Z) = \text{tr}(Y^T X^T Z - X^T Y^T Z) \\ &= \text{tr}(Y X Z - X Y Z) = \text{tr}(Y X Z - Y Z X) \\ &= -\text{tr}(Y^T X Z - Y^T Z X) = -\text{tr}(Y^T [X, Z]) \\ &= -\langle Y, [X, Z] \rangle_e. \end{aligned}$$

Now apply #4.