

# Math 621 HW #4 – Outline of Solution

Note Title

2/19/2018

1. (a)  $\Rightarrow$ : If  $E$  is trivial and  $\varphi: E \rightarrow M \times \mathbb{R}^n$  is a bundle isomorphism, then define  $\sigma_i$  by  $\sigma_i(x) = \varphi^{-1}(x, e_i)$  where  $e_1, \dots, e_n$  is the standard basis of  $\mathbb{R}^n$ .

$\Leftarrow$ : Define  $\Psi: M \times \mathbb{R}^n \rightarrow E$  by  $\Psi(x, x_1, \dots, x_n) = \sum x_i \sigma_i(x)$ .

This is clearly smooth and bijective; we need it to be a diffeomorphism. Work locally on  $U \subset M$  with  $\pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^n$  diffeo.

Then the composition

$$U \times \mathbb{R}^n \xrightarrow{\Psi} \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^n$$

Sends  $(x, v)$  to  $(x, A_x v)$  where  $A_x: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is linear.

The differential matrix for this map is of the form

$$\begin{bmatrix} I & 0 \\ 0 & A_x \end{bmatrix}, \text{ which is invertible. Thus } h \circ \Psi \text{ is a diffeo, so}$$

$\Psi$  is a diffeo.

(b) Write Möbius strip as  $M = ([0, 1] \times \mathbb{R} / (0, y) \sim (1, -y))$ ,

with the projection map  $\pi: M \rightarrow S^1 = ([0, 1] / 0 \sim 1)$ .

Then  $S^1 = V_1 \cup V_2$ ,  $V_1 = (0, 1)$ ,  $V_2 = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] / (0 \sim 1)$ .

local trivialization for  $M$ :

$$\begin{cases} \varphi_1: \pi^{-1}(V_1) \xrightarrow{\cong} V_1 \times \mathbb{R} & \text{identity map} \\ \varphi_2: \pi^{-1}(V_2) \xrightarrow{\cong} V_2 \times \mathbb{R} & \varphi_2(x, y) = \begin{cases} (x, y) & 0 \leq x < \frac{1}{2} \\ (x, -y) & \frac{1}{2} < x \leq 1 \end{cases} \end{cases}$$

If  $M$  were a trivial bundle, then it would have a nowhere vanishing section  $s: S^1 \rightarrow M$ . Over  $V_1$ , we have  $\varphi_1(s(x)) = (x, f(x))$

for some  $f$  with either  $f > 0$  or  $f < 0$ . Then over  $V_2$ , either

$\varphi_2(s(x)) > 0$  for  $0 < x < \frac{1}{2}$  and  $\varphi_2(s(x)) < 0$  for  $\frac{1}{2} < x < 1$ ,

or vice versa; either way implies  $\varphi_2(s(0)) = 0$  so  $s$  vanishes at 0.

(c) Extend a basis of  $g_1$  to a basis of left invariant vector fields on  $G_1$  and use (a).

2. (a) i, ii clear.

$$\begin{aligned}\text{iii: } \frac{d}{dt} \Big|_{t=0} \varphi_t^*(S \otimes T) &= \frac{d}{dt} \Big|_{t=0} (\varphi_t^* S) \otimes (\varphi_t^* T) \\ &= \left( \frac{d}{dt} \Big|_{t=0} \varphi_t^* S \right) \otimes T + S \otimes \left( \frac{d}{dt} \Big|_{t=0} \varphi_t^* T \right) \\ &= (\mathcal{L}_X S) \otimes T + S \otimes (\mathcal{L}_X T).\end{aligned}$$

iv: Differentiate  $\varphi_t^*(C_{ij}(S)) = C_{ij}(\varphi_t^* S)$ :

$$\mathcal{L}_X C_{ij}(S) = \frac{d}{dt} \Big|_{t=0} \varphi_t^*(C_{ij}(S)) = \frac{d}{dt} \Big|_{t=0} C_{ij}(\varphi_t^* S) = C_{ij} \left( \frac{d}{dt} \Big|_{t=0} \varphi_t^* S \right) = C_{ij} \mathcal{L}_X S.$$

important to note: here we use the chain rule and the fact that  $C_{ij}$  is linear,  
so the differential of  $C_{ij}$  is  $C_{ij}'$ .

(b) For  $p \in U$ , let  $V$  be a neighborhood of  $p$  with  $\bar{V} \subset U$ ;

then there is a smooth function  $f \in C^\infty(M)$  such that

$$f = \begin{cases} 1 & \text{inside } V \\ 0 & \text{outside } V \end{cases}. \quad \text{Then if } S=T \text{ on } U, \text{ then}$$

$$\begin{aligned}fS=fT \text{ on } M &\Rightarrow P_X(fS)=P_X(fT) \\ &\Rightarrow (Xf)S+fP_XS=(Xf)T+fP_XT.\end{aligned}$$

$$\text{At } p, Xf(p)=0 \Rightarrow P_XS(p)=P_XT(p).$$

3. (a) Just need  $\mathcal{L}_X$  to be well-defined locally. It's well-defined on functions and on  $\frac{\partial}{\partial x_i}$  by (i), (ii). By (iii) and (iv), we have

$$\begin{aligned}0 &= \mathcal{L}_X(S_{ij}) = \mathcal{L}_X(C_{ij}(\frac{\partial}{\partial x_i} \otimes dx_j)) \\ &= C_{ii} \mathcal{L}_X(\frac{\partial}{\partial x_i} \otimes dx_j) \\ &= C_{ii} \left( (\mathcal{L}_X \frac{\partial}{\partial x_i}) \otimes dx_j \right) + C_{ii} \left( \frac{\partial}{\partial x_i} \otimes (\mathcal{L}_X dx_j) \right) \\ &= C_{ii} \left( (\mathcal{L}_X \frac{\partial}{\partial x_i}) \otimes dx_j \right) + \mathcal{L}_X dx_j \left( \frac{\partial}{\partial x_i} \right).\end{aligned}$$

This determines  $\mathcal{L}_X dx_j$  since it determines  $(\mathcal{L}_X dx_j)(\frac{\partial}{\partial x_i})$  for all  $i$ . Finally,  $\mathcal{L}_X$  is determined for all tensors by (iii). (see next page for alternate solution)

3.(a) Alternate solution if you don't want to use locality:

cont'd.

again it suffices to show  $\mathcal{L}_X$  is uniquely determined on  $(0,1)$  tensors. Let  $\omega \in \Gamma(T^*M)$ . Then for any  $Y \in \text{Vect}(M) = \Gamma(T_0 M)$ ,

$$\begin{aligned} (\mathcal{L}_X \omega)(Y) &= c(Y \otimes \mathcal{L}_X \omega) = c(\mathcal{L}_X(Y \otimes \omega) - (\mathcal{L}_X Y) \otimes \omega) \\ &= \mathcal{L}_X c(Y \otimes \omega) - \omega(\mathcal{L}_X Y) \\ &= X \omega(Y) - \omega([X, Y]); \end{aligned}$$

this uniquely determines  $\mathcal{L}_X \omega \in \Gamma(T^*M)$ .

(b) Consider  $S = S_1 \otimes \dots \otimes S_g$ . Then

$$\mathcal{L}_X S = \sum S_i \otimes \dots (\mathcal{L}_X S_i) \dots \otimes S_g.$$

Assuming the given formula is true for  $g=1$ , we get

$$\begin{aligned} (\mathcal{L}_X S)(Y_1, \dots, Y_g) &= \sum S_i(Y_i) \dots (\mathcal{L}_X S_i)(Y_i) \dots S_g(Y_g) \\ &= \sum S_i(Y_i) \dots (X(S_i(Y_i)) - S_i([X, Y_i])) \dots S_g(Y_g) \\ &= X(S_1(Y_1) \dots S_g(Y_g)) - \sum S_i(Y_i) \dots S_i([X, Y_i]) \dots S_g(Y_g) \\ &= X(S_1(Y_1) \dots S_g(Y_g)) - S(Y_1, \dots, [X, Y_1], \dots, Y_g) \end{aligned}$$

as desired. Thus it suffices to show that if  $S = (0,1)$  tensor, then

$$(\mathcal{L}_X S)(Y) = X S(Y) - S([X, Y]).$$

Part  $\mathcal{L}_X S(Y) = C_{11}(Y \otimes \mathcal{L}_X S)$

$$\begin{aligned} &= C_{11}(\mathcal{L}_X(Y \otimes S) - (\mathcal{L}_X Y) \otimes S) \\ &= \mathcal{L}_X C_{11}(Y \otimes S) - S(\mathcal{L}_X Y) \\ &= X(S(Y)) - S([X, Y]). \end{aligned}$$

$g$  times

Alternate solution: note  $S(Y_1, \dots, Y_g) = \widetilde{C_{11}} \dots \widetilde{C_{11}}(Y_1 \otimes \dots \otimes Y_g \otimes S)$ , so

$$\begin{aligned} X S(Y_1, \dots, Y_g) &= \mathcal{L}_X C_{11}^g(Y_1 \otimes \dots \otimes Y_g \otimes S) = C_{11}^g \mathcal{L}_X(Y_1 \otimes \dots \otimes Y_g \otimes S) \\ &= C_{11}^g \left( \sum Y_i \otimes \dots (\mathcal{L}_X Y_i) \dots \otimes Y_g \otimes S + Y_1 \otimes \dots \otimes Y_g \otimes \mathcal{L}_X S \right) \\ &= \sum S(Y_1, \dots, [X, Y_i], \dots, Y_g) + (\mathcal{L}_X S)(Y_1, \dots, Y_g). \end{aligned}$$