1. (a) Check that \( \psi_x [X,Y] (q) (\psi_t (p)) = X (Y (\psi_t (p))) (p) - Y (X (\psi_t (p))) (p) \)

(b) The answer is \( \psi_t \psi_0 \psi_t^{-1} \).

We need \( \frac{d}{dt} \psi_t \psi_0 \psi_t^{-1} (p) = (\psi_x X) (\psi_t \psi_0 \psi_t^{-1} (p)) \) for all \( t \).

Note

\[
\frac{d}{dt} \bigg|_{t=0} \psi_t \psi_t^{-1} (p) = \psi_x \left( \frac{d}{dt} \bigg|_{t=0} \psi_t (\psi_t^{-1} (p)) \right)
= \psi_x (X (\psi_t^{-1} (p)))
= (\psi_x X) (\psi_t^{-1} (p)).
\]

For arbitrary \( t \),

\[
\frac{d}{dt} \bigg|_{t=t_0} \psi_t \psi_t^{-1} (p) = \frac{d}{dt} \bigg|_{t=t_0} (\psi_{t-t_0} \psi_t^{-1} (p)) (\psi_{t-t_0} \psi_t^{-1} (p))
= \frac{d}{dt} \bigg|_{t=0} (\psi_{t-t_0} \psi_t^{-1} (p)) (\psi_{t-t_0} \psi_t^{-1} (p))
= (\psi_x X) (\psi_{t-t_0} \psi_t^{-1} (p)).
\]

As desired.

(c) First assume that \( \psi_t \)'s commute.

From (b), the time flow of \( (\psi_t)_x Y \) is \( \psi_t \psi_0 \psi_t^{-1} = \psi_t \).

Since this is independent of \( t \), \( (\psi_t)_x Y \) is independent of \( t \),
so \( [X,Y] = \frac{d}{dt} \bigg|_{t=0} (\psi_t)_x Y = 0 \).

Now assume \( [X,Y] = 0 \). Then

\[
0 = \psi_t Y = \frac{d}{dt} \bigg|_{t=t_0} ((\psi_t)_x Y) (p) = \frac{d}{dt} \bigg|_{t=t_0} (\psi_t)_x (Y (\psi_t (p)))
\]

It follows that for any \( t_0 \),

\[
\frac{d}{dt} \bigg|_{t=t_0} ((\psi_t)_x Y) (p) = \frac{d}{dt} \bigg|_{t=t_0} (\psi_t)_x (Y (\psi_t (p)))
\]

Substitute \( t 

\[
= \frac{d}{dt} \bigg|_{t=t_0} (\psi_{t-t_0} \psi_t^{-1}) (\psi_t)_x Y (\psi_{t-t_0} \psi_t^{-1} (p))
= (\psi_{t-t_0}) (\psi_t)^{-1} \frac{d}{dt} \bigg|_{t=t_0} (\psi_t)_x Y (\psi_{t-t_0} \psi_t^{-1} (p))
= (\psi_{t-t_0}) [X,Y] (\psi_{t-t_0} \psi_t^{-1} (p))
= 0.
\]
1. (c) cont'd. Thus \((\varphi_t)^* Y\) is independent of \(t\).

It follows that the time \(s\) flow of \((\varphi_t)^* Y\) is independent of \(t\).
But from (a), this is the map \(\varphi_t^* \gamma_s \circ \varphi_t\). At \(t = 0\),
this is \(\gamma_s\), thus \((\varphi_t)^* \gamma_s \circ \varphi_t = \gamma_s\), as desired.

2. (a) If \(\gamma_x(t)\) is defined on some interval \((-e, e)\) containing \(0\),
then for any \(g \in G\), \(L_g \gamma_x(t)\) is defined on the same interval
and is an integral curve for \(L_g^* \dot{\gamma}_x = \dot{\gamma}_x\). In particular,
\(L_{\gamma_x(0)} \gamma_x(t)\) defines \(\gamma_x(t+e)\) for \((e, e)\), so \(\gamma_x\)
is defined on \((t_0-e, t_0+e)\). Continue in this way to extend to all of \(\mathbb{R}\).

The same argument shows \(\gamma_x(t_0) \gamma_x(t) = L_{\gamma_x(t_0)} \gamma_x(t) = \gamma_x(t+t_0)\).

(b) From above, \(L_g \gamma_x(t) = \gamma_{L_g x}(t)\).

(c) If \(G\) is abelian, then for any \(x, y \in g\), \([x, y] = 0 \Rightarrow \gamma_x \gamma_y = \gamma_y \gamma_x\) \(\forall s \in (0, e)\)
(from (c)) \(\Rightarrow \exp(sx) \exp(sy) = \exp(sy) \exp(sx)\) (from (c))
\(\Rightarrow \exp x, \exp y\) commute. Since \(\exp x\) generates \(G\),
\(G\) is abelian.

If \(G\) is abelian, then the same argument shows if \(x, y \in \mathbb{R}\),
\(\gamma_x \gamma_y = \gamma_y \gamma_x \forall s \Rightarrow [x, y] = 0 \Rightarrow \gamma x y\) is abelian.

3. (a) \(\Rightarrow:\) If \(E\) is trivial and \(\varphi: E \to \mathbb{M} \times \mathbb{R}^n\) is a bundle isomorphism,
then define \(\sigma_t\) by \(\sigma_t(x) = \varphi^{-1}(x, e_t)\) \((e_1, \ldots, e_n)\) is the
standard basis of \(\mathbb{R}^n\).

\(\Leftarrow:\) Define a map \(\gamma: \mathbb{M} \times \mathbb{R}^n \to E\) by \(\gamma(x, x_1, \ldots, x_n) = \sum x_i \sigma_i(x)\).

This is clearly smooth and bijective. We need it to be a diffeomorphism.

Work locally: \(U \subset M, \pi^{-1}(U) \to \mathbb{M} \times \mathbb{R}^n\) diffeo (with \(\pi: E \to M\)).
Then the composition \(\gamma: U \times \mathbb{R}^n \to \pi^{-1}(U) \to U \times \mathbb{R}^n\)
is a diffeomorphism with matrix \(\frac{\partial \gamma}{\partial (x, v)}\), which is invertible.

Sends \((x, v)\) to \((x, Av)\) where \(A: \mathbb{R}^n \to \mathbb{R}^n\) is linear. The differential
matrix for this map is of the form \(\begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}\), which is invertible.

Thus \(\gamma\) is a diffeo, so \(\gamma\) is a diffeo.
3. (b) \( M = \mathbb{Q}_1 \times \mathbb{R} / (0,1) \times (1,-1) \); \( S^1 = [0,1] / 0\cdot1 \).

There's an obvious map \( \pi : M \to S^1 \).

\[ S^1 \cong V_1 \cup V_2, \quad V_1 = (0,1), \quad V_2 = [0,1) \cup (\frac{1}{2},1] / 0\cdot1. \]

\( \varphi_1 : \pi^{-1}(V_1) \to V_1 \times \mathbb{R} \) identity map

\( \varphi_2 : \pi^{-1}(V_2) \to V_2 \times \mathbb{R} \)

\[ \varphi_2(x,y) = \begin{cases} (x,y) & \text{if } 0 < x < \frac{1}{2} \\ (x,y) & \text{if } \frac{1}{2} < x < 1 \end{cases} \]

If this were trivial, it would have a nowhere zero section \( s : S^1 \to M \).

Over \( V_1 \), we get \( \varphi_1(s(x)) = (x, f(x)) \) for some \( f \) with either \( f > 0 \) or \( f < 0 \).

Case (a) Without loss of generality, \( f > 0 \). Then over \( V_2 \), \( f(x) = \frac{1}{2} \).

Case (b) \( f(x) = -\frac{1}{2} \).

This contradicts Intermediate Value Theorem over \( V_2 \).

(C) Extend a basis of \( \mathfrak{g}_1 \) to a left invariant vector fields on \( G \).

\( \mathfrak{g}_1 \) might come down to being if \( \mathfrak{g}_1 \) is abelian and use (c).