

Math 621 HW 2 - Outline of Solutions

Note Title

2/2/2018

1. (a) Extend δ to $C^\infty(M)$ by $\delta(f) = \delta(f-f(p))$ where $f-f(p) \in \mathcal{F}_p$.

(Note if δ is a derivation then this must hold since then $\delta(c)=0$ for $c=\text{const.}$)

This δ is a derivation: if $f, g \in C^\infty(M)$ then

$$\begin{aligned}\delta(fg) &= \delta((fg-f(p))g(p)) = \delta((f-f(p))g(p) + f(p)(g-g(p)) + (f-f(p))(gg(p))) \\ &= \delta(f)g(p) + f(p)\delta(g) + 0.\end{aligned}$$

(b) Define $\{\text{derivations}\} \rightarrow (\mathcal{F}_p/\mathcal{F}_p^2)^*$ by sending δ to

$$\delta|_{\mathcal{F}_p} : \mathcal{F}_p \rightarrow \mathbb{R}.$$

This is well-defined since if $f, g \in \mathcal{F}_p$

$$\text{then } \delta(fg) = 0 \text{ by Leibniz, so } \delta|_{\mathcal{F}_p^2} = 0.$$

The map is surjective by (a) and injective by uniqueness in (a).

2. If we show that ϕ is surjective, then it's an isomorphism

by dimension counting.

Let $v_M \in T_p M$ and $v_N \in T_q N$, and let γ_M, γ_N be curves in M, N with $\gamma_M(0) = p, \gamma_N(0) = q$ such that $\gamma'_M(0) = v_M, \gamma'_N(0) = v_N$.

Then we can define a curve $\gamma = (\gamma_M, \gamma_N)$ in $M \times N$,

and $\gamma_M = \pi_M \circ \gamma, \gamma_N = \pi_N \circ \gamma$. Then by definition of differential,

$$(d\pi_M)_{(q,p)} [\gamma] = [\gamma_M] = v_M$$

equivalence classes in the tangent space

and $(d\pi_N)_{(p,q)} [\gamma] = [\gamma_N] = v_N$; so if we define $v = \gamma'(0) \in \tilde{T}_{(p,q)}(M \times N)$

then $d\pi_M(v) = v_M, d\pi_N(v) = v_N$. This proves surjectivity.

Another solution: let $U_1 \xrightarrow{f_1} V_1, U_2 \xrightarrow{f_2} V_2$ be charts for $p \in M, q \in N$; then

$U_1 \times U_2 \xrightarrow{F=(f_1, f_2)} V_1 \times V_2$ is a chart for $(p, q) \in M \times N$. Then $F^{-1} \circ \pi_M \circ F : U_1 \times U_2 \rightarrow U_1$

is projection, so in coordinates, $d\pi_M = [I \ 0]$. Similarly $d\pi_N = [0 \ I]$

and so $(d\pi_M, d\pi_N)$ is the identity matrix in coordinates.

3. Let $(F_\alpha, U_\alpha, V_\alpha)$, $(F_\beta, U_\beta, V_\beta)$ be overlapping charts on M , giving rise to charts $(\tilde{F}_\alpha, U_\alpha \times \mathbb{R}^n, \tilde{V}_\alpha)$, $(\tilde{F}_\beta, U_\beta \times \mathbb{R}^n, \tilde{V}_\beta)$ on TM . If we have coords (x_1, \dots, x_n) on U_α , (y_1, \dots, y_n) on U_β

$$\rightarrow (x_1, \dots, x_n, v_1, \dots, v_n) \text{ on } U_\alpha \times \mathbb{R}^n, (y_1, \dots, y_n, w_1, \dots, w_n) \text{ on } U_\beta \times \mathbb{R}^n$$

then

$$(y_1, \dots, y_n, w_1, \dots, w_n) = ((F_\beta^{-1} \circ F_\alpha)(x_1, \dots, x_n), d(F_\beta^{-1} \circ F_\alpha)_{(x_1, \dots, x_n)}(v_1, \dots, v_n))$$

and the Jacobian for this map is of the form

$$\begin{bmatrix} d(F_\beta^{-1} \circ F_\alpha) & 0 \\ * & d(F_\beta^{-1} \circ F_\alpha) \end{bmatrix}$$

which has determinant $(\det d(F_\beta^{-1} \circ F_\alpha))^2 > 0$.

Thus $\{\tilde{F}_\alpha, U_\alpha \times \mathbb{R}^n, \tilde{V}_\alpha\}$ is an oriented atlas for TM .

4. First note that the functions $r^\alpha x_1$ and $r^\alpha x_2$ extend continuously to $(x_1, x_2) = (0, 0)$ iff $\alpha > -1$, and extend smoothly iff $\alpha \geq 0$: if $-1 < \alpha < 0$ then $\frac{\partial}{\partial x_1}|_{(0,0)} \frac{x_1}{(x_1^2 + x_2^2)^{\alpha/2}} = \infty$. Thus the vector field $X := r^\alpha (x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2})$ on $\mathbb{R}^2 \setminus \{0\}$ extends smoothly to \mathbb{R}^2 iff $\alpha \geq 0$: on S^2 , this is the same as extending smoothly from $S^2 \setminus \{N, S\}$ to the South pole.

To determine when X extends smoothly to the North pole, we need to write X in the other coordinate chart (y_1, y_2) , $y_1 = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$, $y_2 = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$:

$$\frac{\partial}{\partial x_1} = \frac{\partial y_1}{\partial x_1} \frac{\partial}{\partial y_1} + \frac{\partial y_2}{\partial x_1} \frac{\partial}{\partial y_2} = \frac{1}{r^4} \left((x_2^2 - x_1^2) \frac{\partial}{\partial x_1} - 2x_1 x_2 \frac{\partial}{\partial x_2} \right)$$

and similarly $\frac{\partial}{\partial x_2} = \frac{1}{r^4} \left(-2x_1 x_2 \frac{\partial}{\partial x_1} + (x_1^2 - x_2^2) \frac{\partial}{\partial x_2} \right)$. Note $r = \sqrt{y_1^2 + y_2^2}$ so $X = r^\alpha (x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}) = r^\alpha \left(-\frac{x_1 x_2^2 - x_1^3}{r^4} \frac{\partial}{\partial y_1} + \frac{-x_1^2 x_2 - x_2^3}{r^4} \frac{\partial}{\partial y_2} \right) = - (y_1 + y_2)^{\frac{\alpha}{2}} (y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2})$.

This extends smoothly to $(y_1, y_2) = (0, 0)$ (which is the North pole) iff $\alpha \leq 0$.

The result follows.