Math 621 Homework 11—“due” Wednesday April 24, but not to be handed in
Spring 2013

Outlines of solutions are posted on the course web page.

1. (a) Let $G$ be a connected Lie group whose Lie algebra $\mathfrak{g}$ has trivial center: that is, the only $X \in \mathfrak{g}$ for which $[X, Y] = 0$ for all $Y \in \mathfrak{g}$ is $X = 0$. Suppose that $G$ has a biinvariant metric. Prove that $G$ and its universal cover are compact.

(b) Use (a) to give another proof that $SL_n(\mathbb{R})$ has no biinvariant metric.

(c) Any compact connected Lie group has a biinvariant metric (see Exercise 7, pp. 46–47 for details). Use this to deduce an important result in Lie theory, “Weyl’s Theorem”: the universal cover of any compact connected semisimple Lie group is compact. (You don’t need to know what “semisimple” means, only that it implies in particular that $\mathfrak{g}$ has trivial center.)

2. Prove that $\mathbb{RP}^2 \times \mathbb{RP}^2$ has no metric with strictly positive sectional curvature. (There’s a related open problem, the Hopf conjecture, that posits that $S^2 \times S^2$ has no metric with strictly positive sectional curvature.)

3. Prove that any even-dimensional complete manifold with constant sectional curvature $K = 1$ is isometric to either $S^n$ with the round metric, or $\mathbb{RP}^n$ with the metric induced from the round metric on $S^n$. (For the solution, see Proposition 4.4, pp. 166–167.)