In this problem set, \( R \langle x_1, x_2, \ldots \rangle \) denotes the free \( R \)-module generated by \( x_1, x_2, \ldots \), and “calculate” means describe as an \( R \)-module (with justification, of course).

1. Let \((K, D)\) be the filtered complex given by

\[
K = \mathbb{Z}\langle x_0, y_0, x_1, y_1 \rangle \\
\mathcal{F}^0(K) = K \\
\mathcal{F}^1(K) = \mathbb{Z}\langle x_1, y_1 \rangle \\
\mathcal{F}^2(K) = 0
\]

\[
D(x_0) = 2y_0 + y_1 \\
D(y_0) = 0 \\
D(x_1) = 2y_1 \\
D(y_1) = 0.
\]

Calculate the spectral sequence \( E_1, E_2, \ldots, E_\infty \). Is \( E_\infty \) isomorphic to \( H(K) \)?

2. Let \((K, D)\) be the filtered complex given by

\[
K = \mathbb{R}\langle x_0, y_0, x_1, y_1, x_2, y_2, \ldots \rangle
\]

with \( D(x_m) = y_m \) and \( D(y_m) = 0 \) for all \( m \), with filtration

\[
\mathcal{F}^0(K) = K, \\
\mathcal{F}^m(K) = \mathbb{R}\langle y_{m-1}, y_m, y_{m+1}, \ldots \rangle, \quad m \geq 1.
\]

Calculate \( H(K) \), and show that the spectral sequence \( E_1, E_2, \ldots \) does not converge to \( H(K) \).

3. Let \((K, D)\) be the filtered complex given by

\[
K = \mathbb{R}\langle x_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, y_n \rangle
\]

with \( D(x_0) = y_1 \), \( D(x_m) = y_{m+1} - y_m \) for \( 1 \leq m \leq n-1 \), and \( D(y_m) = 0 \) for all \( m \), with filtration

\[
\mathcal{F}^0(K) = K, \\
\mathcal{F}^m(K) = \mathbb{R}\langle x_m, y_m, x_{m+1}, y_{m+1}, \ldots, x_{n-1}, y_{n-1}, y_n \rangle, \quad 1 \leq m \leq n-1, \\
\mathcal{F}^n(K) = \mathbb{R}\langle y_n \rangle, \\
\mathcal{F}^{n+1}(K) = 0.
\]

Calculate \( H(K) \) and the spectral sequence \( E_1, E_2, \ldots, E_\infty \).

(Hint: you should find that the spectral sequence converges to \( E_\infty = H(K) \) beginning at \( E_{n+1} \). This is an example where the differential \( d_n \) is nonzero, even though each term in the original differential \( D \) only changes filtration level by at most 1.)
4. Let \((K, D)\) be the complex given by

\[
K = \mathbb{R}\langle u, v, w, x, y, z \rangle \\
D(u) = w + x + z \\
D(v) = w + x \\
D(w) = y \\
D(x) = -y \\
D(y) = D(z) = 0.
\]

(a) Calculate \(H(K)\).

(b) For the filtration on \(K\) given by

\[
\mathcal{F}^0(K) = K, \\
\mathcal{F}^1(K) = \mathbb{R}\langle v, w, x, y, z \rangle, \\
\mathcal{F}^2(K) = \mathbb{R}\langle x, y \rangle, \\
\mathcal{F}^3(K) = 0,
\]

calculate the spectral sequence \(E_1, E_2, \ldots, E_\infty\).

(c) For the filtration on \(K\) given by

\[
\mathcal{F}^0(K) = K, \\
\mathcal{F}^1(K) = \mathbb{R}\langle w, x, y, z \rangle, \\
\mathcal{F}^2(K) = \mathbb{R}\langle y, z \rangle, \\
\mathcal{F}^3(K) = 0,
\]

calculate the spectral sequence \(E_1, E_2, \ldots, E_\infty\).