Math 262 HW#9

1. $$\big(\delta(\omega\cup\tau)\big)_{\alpha_0\cdots\alpha_{k_1+k_2+1}} = (-1)^{l_1k_2}\Big[\sum_{i\le k_1+1}(-1)^i\omega_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{k_1+1}}\wedge\tau_{\alpha_{k_1+1}\cdots\alpha_{k_1+k_2+1}} + \sum_{i\ge k_1}(-1)^i\omega_{\alpha_0\cdots\alpha_{k_1}}\wedge\tau_{\alpha_{k_1}\cdots\hat{\alpha}_i\cdots\alpha_{k_1+k_2+1}}\Big]$$

Note there are two extraneous cancelling terms ($i=k_1+1$ in the 1st sum, $i=k_1$ in the 2nd term)

$$= (-1)^{l_1k_2}\Big[(\delta\omega)_{\alpha_0\cdots\alpha_{k_1+1}}\wedge\tau_{\alpha_{k_1+1}\cdots\alpha_{k_1+k_2+1}} + (-1)^{k_1}\omega_{\alpha_0\cdots\alpha_{k_1}}\wedge(\delta\tau)_{\alpha_{k_1}\cdots\alpha_{k_1+k_2+1}}\Big]$$

$$= \Big[(\delta\omega)\cup\tau + (-1)^{k_1+l_1}(\omega\cup\delta\tau)\Big]_{\alpha_0\cdots\alpha_{k_1+k_2+1}}$$

and $$d(\omega\cup\tau)_{\alpha_0\cdots\alpha_{k_1+k_2}} = (-1)^{l_1k_2}\Big[d\omega_{\alpha_0\cdots\alpha_{k_1}}\wedge\tau_{\alpha_{k_1}\cdots\alpha_{k_1+k_2}} + (-1)^{l_1}\omega_{\alpha_0\cdots\alpha_{k_1}}\wedge d\tau_{\alpha_{k_1}\cdots\alpha_{k_1+k_2}}\Big]$$

$$= \Big[(-1)^{k_2}(d\omega)\cup\tau + (-1)^{l_1}\omega\cup(d\tau)\Big]_{\alpha_0\cdots\alpha_{k_1+k_2}}$$

$$\Rightarrow D(\omega\cup\tau) = \delta(\omega\cup\tau) + (-1)^{k_1+k_2}d(\omega\cup\tau) = (\delta\omega)\cup\tau + (-1)^{k_1+l_1}\omega\cup(\delta\tau) + (-1)^{k_1}(d\omega)\cup\tau + (-1)^{k_1+k_2+l_1}\omega\cup(d\tau)$$

$$= (D\omega)\cup\tau + (-1)^{k_1+l_1}\omega\cup(D\tau).$$

2. $$(\delta K\omega)_{\beta_0\cdots\beta_k} = \sum_j(-1)^j(K\omega)_{\beta_0\cdots\hat{\beta}_j\cdots\beta_k} = \sum_{i<j}(-1)^{i+j}\omega_{\varphi(\beta_0)\cdots\varphi(\beta_i)\psi(\beta_i)\cdots\widehat{\psi(\beta_j)}\cdots\psi(\beta_k)}$$
$$+\sum_{i>j}(-1)^{i+j+1}\omega_{\varphi(\beta_0)\cdots\widehat{\varphi(\beta_j)}\cdots\varphi(\beta_i)\psi(\beta_i)\cdots\psi(\beta_k)}$$

$$(K\delta\omega)_{\beta_0\cdots\beta_k} = \sum_{i=0}^{k}(-1)^i(\delta\omega)_{\varphi(\beta_0)\cdots\varphi(\beta_i)\psi(\beta_i)\cdots\psi(\beta_k)}$$

$$= \Big(\sum_{i>j}(-1)^{i+j}(\delta\omega)_{\varphi(\beta_0)\cdots\widehat{\varphi(\beta_j)}\cdots\varphi(\beta_i)\psi(\beta_i)\cdots\psi(\beta_k)} + \sum_{i=0}^{k}\omega_{\varphi(\beta_0)\cdots\varphi(\beta_{i-1})\psi(\beta_i)\cdots\psi(\beta_k)}\Big)$$

$$+\Big(\sum_{i<j}(-1)^{i+j+1}\omega_{\varphi(\beta_0)\cdots\varphi(\beta_i)\psi(\beta_i)\cdots\widehat{\psi(\beta_j)}\cdots\psi(\beta_k)} - \sum_{i=0}^{k}\omega_{\varphi(\beta_0)\cdots\varphi(\beta_i)\psi(\beta_{i+1})\cdots\psi(\beta_k)}\Big)$$

telescoping series

$$= -(\delta K\omega)_{\beta_0\cdots\beta_k} + \omega_{\psi(\beta_0)\cdots\psi(\beta_k)} - \omega_{\varphi(\beta_0)\cdots\varphi(\beta_k)}$$

$$= (-\delta K\omega + \psi^{\#}\omega - \varphi^{\#}\omega)_{\beta_0\cdots\beta_k}.$$

3. $\delta : \underset{\mathbb{Z}^3}{\underset{\parallel}{C^0(\mathcal{U},\mathcal{F})}} \longrightarrow \underset{\mathbb{Z}^3}{\underset{\parallel}{C^1(\mathcal{U},\mathcal{F})}}$ is given by

$\delta(f_0, f_1, f_2) = ((\delta f)_{01}, (\delta f)_{02}, (\delta f)_{12})$ where

$(\delta f)_{01} = \rho^1_{01} f_1 - \rho^0_{01} f_0 = f_1 - f_0$

$(\delta f)_{02} = \rho^2_{02} f_2 - \rho^0_{02} f_0 = -f_2 - f_0$

$(\delta f)_{12} = \rho^2_{12} f_2 - \rho^1_{12} f_1 = f_2 - f_1$

and $\check{H}^0(\mathcal{U},\mathcal{F}) = \ker \delta = 0$

$\check{H}^1(\mathcal{U},\mathcal{F}) = \operatorname{coker} \delta = \mathbb{Z}/2$ $\quad (\operatorname{im} \delta = \{(x,y,z) \in \mathbb{Z}^3 \mid x+y+z \equiv 0 \pmod 2\})$.

$(\check{H}^k(\mathcal{U},\mathcal{F}) = 0$ for $k \geq 2)$.