Math 262 HW#8

1. Define $\varphi: K^{i,j} \to \tilde{K}^{j,i} (= K^{i,j})$ by $\varphi(x) = (-1)^{ij} x$; this extends to $\varphi: K \to \tilde{K}$, and sends the single grading on K to the single grading on $\tilde{K}$. Then on $K^{i,j}$, $\varphi\delta = (-1)^j \tilde{d}\varphi$ and $\varphi d = (-1)^i \tilde{\delta}\varphi$, so $\varphi D = \tilde{D}\varphi$ and $\varphi: (K, D) \to (\tilde{K}, \tilde{D})$ is the desired chain isomorphism.

2. Say α is in l-th position, β in m-th in the word $\alpha_0 \cdots \alpha \cdots \beta \cdots \alpha_{k+1}$. $(\omega \in C^k(\mathcal{U}, \Omega^*))$
Then

$$(\delta\omega)_{\alpha_0 \cdots \alpha \cdots \beta \cdots \alpha_{k+1}} = \sum (-1)^i \omega_{\cdots \hat{\alpha}_i \cdots \alpha \cdots \beta \cdots} + \sum (-1)^i \omega_{\cdots \alpha \cdots \hat{\alpha}_i \cdots \beta \cdots} + \sum (-1)^i \omega_{\cdots \alpha \cdots \beta \cdots \hat{\alpha}_i \cdots}$$
$$+ (-1)^l \omega_{\cdots \hat{\alpha} \cdots \beta \cdots} + (-1)^m \omega_{\cdots \alpha \cdots \hat{\beta} \cdots}$$

and $-(\delta\omega)_{\alpha_0 \cdots \beta \cdots \alpha \cdots \alpha_{k+1}}$ is the same sum, but with $-(-1)^m \omega_{\cdots \beta \cdots \hat{\alpha} \cdots} - (-1)^l \omega_{\cdots \hat{\beta} \cdots \alpha \cdots}$ instead of the last two terms. But

$$\omega_{(\cdots)_1 \hat{\alpha} \underbrace{(\cdots)_2}_{l-k-1 \text{ indices}} \beta (\cdots)_3} = (-1)^{l-k-1} \omega_{(\cdots)_1 \beta (\cdots)_2 \hat{\alpha} (\cdots)_3}$$

and similarly $\omega_{\cdots \alpha \cdots \hat{\beta} \cdots} = (-1)^{l-k-1} \omega_{\cdots \hat{\beta} \cdots \alpha \cdots}$, and the result follows.

3. (a) If $X =$ disjoint union of $\mathbb{R}^n$'s, then

$$0 \to (\text{loc. const. fns. on } X) \to \Omega^0(X) \to \Omega^1(X) \to \cdots \to \Omega^n(X) \to 0$$

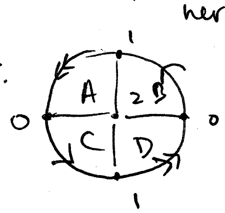
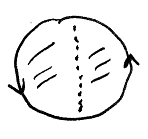
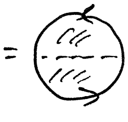
is exact by Poincaré Lemma; thus if $\mathcal{U} =$ decent cover,

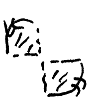
$$0 \to \prod(\text{loc. const. fns. on } U_{\alpha_0 \cdots \alpha_k}) \to \prod \Omega^0(U_{\alpha_0 \cdots \alpha_k}) \to \prod \Omega^1(U_{\alpha_0 \cdots \alpha_k}) \to \cdots$$

is exact. Thus the same argument as for good covers shows that

$$H^*_{DR}(M) = H^*(C^*(\mathcal{U}, \Omega^*)) = \check{H}^*(\mathcal{U}, \mathbb{R}).$$

(b) Decent cover $\underset{\text{nerve}}{\longleftrightarrow}$ Δ-complex.

$\mathbb{RP}^2$:  $U_0 =$ (open star of 0) $=$  ; $U_1 =$  ; $U_2 =$ 

$U_{01} =$  $= AD \amalg BC$ where $AD =$  etc

$U_{02} = AC \amalg BD$; $U_{12} = AB \amalg CD$; $U_{012} = A \amalg B \amalg C \amalg D$.

3(b) Cont'd.

We have $C^0(\mathcal{U},\mathbb{R}) \cong \mathbb{R}^3$ with coords. (f_0, f_1, f_2)

$\uparrow$ loc const fn on U_0 etc

$C^1(\mathcal{U},\mathbb{R}) \cong \mathbb{R}^6$ with coords $(g_{01}^{AD}, g_{01}^{BC}, g_{02}^{AC}, g_{02}^{BD}, g_{12}^{AB}, g_{12}^{CD})$

$\uparrow$ loc const fn on $AD \subset U_{01}$ etc.

$C^2(\mathcal{U},\mathbb{R}) \cong \mathbb{R}^4$ with coords (h^A, h^B, h^C, h^D)

and $\delta(f_0, f_1, f_2) = (g_{01}^{AD}, \ldots)$ where $g_{01}^{AD} = g_{01}^{BC} = f_1 - f_0$, $g_{02}^{AC} = g_{02}^{BD} = f_2 - f_0$, $g_{12}^{AB} = g_{12}^{CD} = f_2 - f_1$,

$\delta(g_{01}^{AD}, \ldots) = (h^A, \ldots)$ where $h^A = g_{12}^{AB} - g_{02}^{AC} + g_{01}^{AD}$ $\quad h^C = g_{12}^{CD} - g_{02}^{AC} + g_{01}^{BC}$

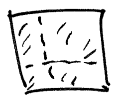
$h^B = g_{12}^{AB} - g_{02}^{BD} + g_{01}^{BC}$ $\quad h^D = g_{12}^{CD} - g_{02}^{BD} + g_{01}^{AD}$.

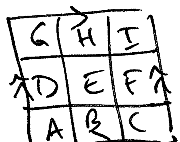
Then $\ker(\delta : C^0(\mathcal{U},\mathbb{R}) \to C^1(\mathcal{U},\mathbb{R})) = \mathbb{R} = \{(x,x,x) \mid x \in \mathbb{R}\}$ while

$\operatorname{im}(\delta : C^1(\mathcal{U},\mathbb{R}) \to C^2(\mathcal{U},\mathbb{R})) = C^2(\mathcal{U},\mathbb{R}) = \mathbb{R}^4$ (linear alg. exercise),

so dimension counting gives $\check{H}^0(\mathcal{U},\mathbb{R}) \cong \mathbb{R}$, $\check{H}^1(\mathcal{U},\mathbb{R}) = \check{H}^2(\mathcal{U},\mathbb{R}) = 0$ and this is $H^*_{DR}(\mathbb{RP}^2)$.

3(c) As some of you pointed out, there's actually a 3-set decent cover of T^2, so we'll use that.

$T^2 =$  $\quad U_0 =$  $\quad U_1 =$  $\quad U_2 =$ 



$U_{01} = A \sqcup BC \sqcup DG \sqcup EFHI$

$U_{02} = ABDE \sqcup CF \sqcup GH \sqcup I$

$U_{12} = ACGI \sqcup BH \sqcup DF \sqcup E$

$U_{012} = A \sqcup B \sqcup C \sqcup D \sqcup E \sqcup F \sqcup G \sqcup H \sqcup I$.

$C^0(\mathcal{U},\mathbb{R}) \cong \mathbb{R}^3$, $C^1(\mathcal{U},\mathbb{R}) \cong \mathbb{R}^{12}$, $C^2(\mathcal{U},\mathbb{R}) \cong \mathbb{R}^9$

and with notation as in (b),

$\delta : C^0 \to C^1$ is given by $g_{01}^{A} = g_{01}^{BC} = g_{01}^{DG} = g_{01}^{EFHI} = f_1 - f_0$, etc

$\delta : C^1 \to C^2$ is given by

$h^A = g_{12}^{ACGI} - g_{02}^{ABDE} + g_{01}^{A}$ $\quad h^D = g_{12}^{DF} - g_{02}^{ABDE} + g_{01}^{DG}$ $\quad h^G = g_{12}^{ACGI} - g_{02}^{GH} + g_{01}^{DG}$

$h^B = g_{12}^{BH} - g_{02}^{ABDE} + g_{01}^{BC}$ $\quad h^E = g_{12}^{E} - g_{02}^{ABDE} + g_{01}^{EFHI}$ $\quad h^H = g_{12}^{BH} - g_{02}^{GH} + g_{01}^{EFHI}$

$h^C = g_{12}^{ACGI} - g_{02}^{CF} + g_{01}^{BC}$ $\quad h^F = g_{12}^{DF} - g_{02}^{CF} + g_{01}^{EFHI}$ $\quad h^I = g_{12}^{ACGI} - g_{02}^{I} + g_{01}^{EFHI}$

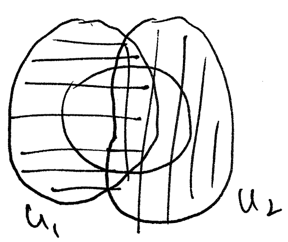
and some linear alg gives

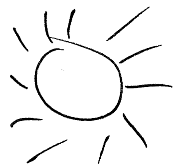
$$H^*_{DR}(T^2) \cong \check{H}^*(\mathcal{U},\mathbb{R}) \cong \begin{cases} \mathbb{R} & * = 0 \text{ or } * = 2 \\ \mathbb{R}^2 & * = 1 \\ 0 & \text{otherwise.} \end{cases}$$

3(d) If S^2 had a decent cover with ≤ 2 sets, then

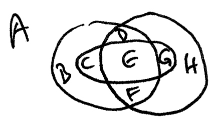
$H^2_{DR}(S^2) \cong \check{H}^2(\mathcal{U},\mathbb{R}) \cong 0$. Thus any decent cover has ≥ 3 sets.

View $S^2 = \mathbb{R}^2 \cup \{\infty\}$, and write



and $U_0 =$ 

$U_{01} = BDF$, $U_{02} = DFH$, $U_{12} = DEF$, $U_{012} = D \amalg F$.



$C^0(\mathcal{U},\mathbb{R}) \cong \mathbb{R}^3$, $C^1(\mathcal{U},\mathbb{R}) \cong \mathbb{R}^3$, $C^2(\mathcal{U},\mathbb{R}^2) \cong \mathbb{R}^2$,

$\delta: C^0 \to C^1$ is given by $g_{01} = f_1 - f_0$, $g_{02} = f_2 - f_0$, $g_{12} = f_2 - f_1$

$\delta: C^1 \to C^2$ is given by $h^D = g_{12} - g_{02} + g_{01}$, $h^F = g_{12} - g_{02} + g_{01}$,

and linear alg gives

$$H^*_{DR}(S^2) \cong \check{H}^*(\mathcal{U},\mathbb{R}) \cong \begin{cases} \mathbb{R} & *=0 \text{ or } *=2 \\ 0 & \text{otherwise.} \end{cases}$$