Math 262 HW #7

1. If $p=2$ then $H^*(M) = \begin{cases} \mathbb{R}^2 & *=0 \\ 0 & \text{otherwise} \end{cases}$ if $M = U \cup V$ and $U \cap V = \phi$;

if $U \cap V \cong \mathbb{R}^n$ then M-V gives $H^*(M) = \begin{cases} \mathbb{R} & *=0 \\ 0 & \text{otherwise} \end{cases}$.

(a) Induction on p ; suppose $M = U \cup V$ where $V \cong \mathbb{R}^n$ and U, $U \cap V$ have good $(p-1)$-set open covers. Then $H^k(U) \oplus H^k(V) = 0$ for $k \geq p-2$ and $H^{k-1}(U \cap V) = 0$ for $k-1 \geq p-2$, so by M-V, $H^k(U \cup V) = 0$ for $k \geq p-1$.

(b) Induction on p again. Note if we have a long exact sequence

$$\cdots \to A^n \to B^n \xrightarrow{f_n} C^n \to A^{n+1} \to \cdots$$

then $A^n \cong \operatorname{coker} f_{n-1} \oplus \ker f_n$, so $\dim A^n \leq \dim C^{n-1} \oplus \dim B^n$, so the total rank of A^* is $\leq (\operatorname{rank} B^*) + (\operatorname{rank} C^*)$, with equality $\Leftrightarrow$ all $f_n = 0$.

Now apply this to $A^n = H^n(M)$, $B^n = H^n(U) \oplus H^n(V)$, $C^n = H^n(U \cap V)$ for U, V as in (a). Note that

$$H^0(U) \oplus \underset{\mathbb{R}}{\underset{\parallel}{H^0(V)}} \to H^0(U \cap V)$$

is the 0 map $\Leftrightarrow U \cap V = \phi$. So if $U \cap V \neq \phi$, then

$$rk(H^*(M)) \leq rk\, H^*(U) + rk\, H^*(V) + rk\, H^*(U \cap V) - 1 \leq 2^{p-2} + 1 + 2^{p-2} - 1 = 2^{p-1},$$

while if $U \cap V = \phi$, then $H^n(M) = H^n(U) \oplus H^n(V)$ so $rk\, H^*(M) = rk\, H^*(U) + 1 \leq 2^{p-2} + 1$.

In both cases $rk\, H^*(M) \leq 2^{p-1}$.

2. (a). If $(\operatorname{supp} \omega) = K_1 \subset M$ and $(\operatorname{supp} \eta) = K_2 \subset N$ then $\operatorname{supp} \pi_1^* \omega = K_1 \times N$ and $\operatorname{supp} \pi_2^* \eta = M \times K_2$, so $\operatorname{supp}(\pi_1^*\omega \wedge \pi_2^*\eta) \subset K_1 \times K_2$ and $\psi(\omega \otimes \eta)$ is compactly supported.

- If $d\omega = d\eta = 0$ then $d\psi(\omega \otimes \eta) = d\pi_1^*\omega \wedge \pi_2^*\eta \pm \pi_1^*\omega \wedge d\pi_2^*\eta = \pi_1^* d\omega \wedge \pi_2^*\eta \pm \pi_1^*\omega \wedge \pi_2^* d\eta = 0$.
- If $d\eta = 0$ and $\omega = d\tau$ then $\psi(\omega \otimes \eta) = \pi_1^* d\tau \wedge \pi_2^*\eta = d\pi_1^*\tau \wedge \pi_2^*\eta = d(\pi_1^*\tau \wedge \pi_2^*\eta) = d\psi(\tau \otimes \eta)$.
- Similarly if $d\omega = 0$ and $\eta = d\tau$ then $\psi(\omega \otimes \eta)$ is exact.

(b) If $\dim M = m$ and $\dim N = n$, then

$$H_c^k(M \times N) \underset{\text{PD}}{\cong} (H^{m+n-k}(M \times N))^* \underset{\text{Künneth}}{\cong} \Big(\bigoplus_{i+j=m+n-k} H^i(M) \otimes H^j(N)\Big)^* \overset{\text{finite-dimensionality}}{\cong} \bigoplus_{i+j=m+n-k} (H^i(M))^* \otimes (H^j(N))^*$$

$$\underset{\text{PD}}{\cong} \bigoplus_{i+j=m+n-k} H_c^{m-i}(M) \otimes H_c^{n-j}(N) \cong \bigoplus_{i+j=k} H_c^i(M) \otimes H_c^j(N).$$

(note need to check $M \times N$ has a finite good cover)

2. (c) Induction on the number of sets in a good cover of M.

If $M \cong \mathbb{R}^n$ then by Poincaré Lemma,

$$H_c^k(\mathbb{R}^n \times N) \cong H_c^{k-n}(N)$$

with $\cong$ given by integration over $\mathbb{R}^n$. Thus if $\omega \in \Omega^n_c(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \omega = 1$,

$$\begin{aligned} \psi : H_c^n(\mathbb{R}^n) \otimes H_c^k(N) &\longrightarrow H_c^{n+k}(\mathbb{R}^n \times N) \xrightarrow{\ \int\ } H_c^k(N) \\ [\omega] \otimes [\eta] &\longmapsto [\pi_1^*\omega \wedge \pi_2^*\eta] \longmapsto [\eta] \end{aligned}$$

So ψ is an isomorphism.

Now suppose Künneth holds for $M = U$, $M = V$, $M = U \cap V$; we want it to hold for $M = U \cup V$. Then M-V for compact support gives row exact sequences

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \bigoplus_{i+j=k} H_c^i(U\cap V)\otimes H_c^j(N) \arrow[r] \arrow[d, "\psi"] \arrow[dr, phantom, "\circlearrowleft"] & \bigoplus_{i+j=k} \left(H_c^i(U)\otimes H_c^j(N)\right)\oplus\left(H_c^i(V)\otimes H_c^j(N)\right) \arrow[r] \arrow[d, "\psi"] \arrow[dr, phantom, "\circlearrowleft"] & \bigoplus_{i+j=k} H_c^i(U\cup V)\otimes H_c^j(N) \arrow[r] \arrow[d, "\psi"] \arrow[dr, phantom, "\circlearrowleft"] & \bigoplus_{i+j=k} H_c^{i+1}(U\cap V)\otimes H_c^j(N) \arrow[r] \arrow[d, "\psi"] & \cdots \\
\cdots \arrow[r] & H_c^k((U\cap V)\times N) \arrow[r] & H_c^k(U\times N)\oplus H_c^k(V\times N) \arrow[r] & H_c^k((U\cup V)\times N) \arrow[r] & H_c^{k+1}((U\cap V)\times N) \arrow[r] & \cdots
\end{tikzcd}$$

If we can show this diagram is commutative, then the result follows from induction and the Five Lemma. The only square that isn't tautological is

$$\begin{tikzcd}
\bigoplus_{i+j=k} H_c^i(U\cup V)\otimes H_c^j(N) \arrow[r, "\tilde{\delta}\otimes \mathrm{id}"] \arrow[d, "\psi"] & \bigoplus_{i+j=k} H_c^{i+1}(U\cap V)\otimes H_c^j(N) \arrow[d, "\psi"] \\
H_c^k((U\cup V)\times N) \arrow[r, "\tilde{\delta}"] & H_c^{k+1}((U\cap V)\times N)
\end{tikzcd}$$

If $\{\rho_U, \rho_V\}$ is a partition of unity subordinate to $\{U, V\}$ and $[\omega]\otimes[\eta] \in \bigoplus H_c^i(U\cup V)\otimes H_c^j(N)$ then $\tilde{\delta}\omega = -d(\rho_U\omega)|_{U\cap V}$, and $\{\underset{\pi_1^*\rho_U}{\underset{\shortparallel}{\rho_U\pi_1}}, \rho_V\pi_1\}$ is a partition of unity subordinate to $\{U\times N, V\times N\}$.

Thus

$$\begin{aligned}
\tilde{\delta}\psi(\omega\otimes\eta) &= \tilde{\delta}(\pi_1^*\omega \wedge \pi_2^*\eta) = -d(\pi_1^*\rho_U\,\pi_1^*\omega\wedge\pi_2^*\eta) = -d(\pi_1^*(\rho_U\omega)\wedge\pi_2^*\eta) \\
&= -d\pi_1^*(\rho_U\omega)\wedge\pi_2^*\eta \pm \pi_1^*(\rho_U\omega)\wedge \underbrace{d\pi_2^*\eta}_{\pi_2^* d\eta = 0} \\
&= -\pi_1^* d(\rho_U\omega)\wedge\pi_2^*\eta \\
&= \psi(-d(\rho_U\omega)\otimes\eta) \\
&= \psi\circ(\tilde{\delta}\otimes\mathrm{id})(\omega\otimes\eta)
\end{aligned}$$

and the square commutes. $\square$